MATHEMATICAL MODELING AND OPTIMIZATION OF BEAM DYNAMICS IN ACCELERATORS

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Abstract

In this paper we treat the problem of beam dynamics optimization as a control theory problem. We consider different mathematical models of optimization. The approach to solving optimization problem for charged particle dynamics in accelerators includes: construction of mathematical model of controlled dynamical process; choice of control functions or parameters of optimization; construction of quality functionals, which allow efficient evaluation of various characteristics of examined controlled motion; analytical representation of the functional variations, which allow to construct various methods of optimization for quality functionals; construction of methods and algorithms of optimization. Problem of statement is considered on the pattern of RFQ channel.

INTRODUCTION

Mathematical methods of modeling and optimization are extensively used in many fields of science and technology. Development of specialized software for various applications is of more and more importance. A special class of tasks attracting attention of numerous researches includes the problems associated with the beam dynamics optimization in accelerator [1–13]. There are not the general methods of accelerating and focusing structures optimization. However as the demand to output beam parameters are progressively increasing it is needed to develop a new approaches and methods to solve these problems. In the paper the different mathematical control models describing beam dynamics are presented. Especially we consider the problems related to charged particles interaction. In this case we investigate the controlled dynamic process described by a system of integro-differential equations. The optimization methods are developed for the different functionals concerned with the quality of beam [3–12]. They are used for solution of various beam dynamics problems in. In particular, we investigate the optimization problem of a radial matching section in RFQ channel. We consider the problem of construction self-consistent distribution for charged particle beam in magnetic field too [14–19].

MATHEMATICAL OPTIMIZATION MODELS

The problem of beam control of interacting particles, which dynamics is described by integro-differential equations, is considered. Let us assume that evolution of particle beam is described by equations

$$dx/dt = f(t,x,u) \tag{1}$$

$$f(t,x,u) = f_1(t,x,u) + \int_{M_{t,u}} f_2(t,x,y_t)\rho(t,y_t)dy_t, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} f(t,x,u) + \rho\,\mathrm{div}_x f(t,x,u) = 0, \tag{3}$$

$$x(0) = x_0 \in M_0, \quad \rho(0,x) = \rho_0(x). \tag{4}$$

Here t is is the time; x is n-vector of phase coordinates; $u = u(t) \in D$ is r-dimensional control vector-function; D is the set of admissible control functions; $\rho = \rho(t,x)$ is the particle distribution density in the phase space; f_1 is n-dimensional vector-function determined by external electromagnetic fields; f_2 is n-dimensional vector-function associated with the particle interactions; the set $M_{t,u}$ is the cross-section of the trajectory set. It is obtained by time shift of the initial set M_0 through solutions of equation (1) with given control $u = u(t)$. The set M_0 is a given set in the phase space, which describes the set of initial states for a charged particle beam at the initial time moment. The function $\rho_0(x)$ is a given function describing the particle distribution density at the moment $t = 0$. The equations (1)-(2) can be considered as Vlasov equations. We meet with these equations if interaction between particles, for example the Coulomb repulsion, is taken into account. Let us introduce a functional

$$I(u) = \int_0^T \int_{M_{t,u}} \varphi(t,x_t,\rho(t,x_t),u)dx_t dt + \int_{M_{T,u}} g(x_T,\rho_T(T,x_T))dx_T \to \min_{u\in D}, \tag{5}$$

characterizing the dynamics of the process. Here φ and g are given non-negative functions, T is fixed. Consider the minimization problem of functional (5). Analizing various systems which are designed for acceleration, focusing and transporting of charged particle beams, it should be noticed that electrical and magnetic fields can be treated in a certain structural and parametric form. Thus certain components and parameters of electromagnetic fields and geometric systems of accelerating or focusing can be taken as control variables. The developed approach can be applied to another kind of functionals:

$$I(u) = \Phi(\mu_{ks}^{(1,1)}, \ldots, \mu_{ks}^{(i,j)}, \ldots, \mu_{ks}^{(n,n)}), \tag{6}$$

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where

$$\mu_{ks}^{(i,j)} = \int_{M_{t,u}} (x_i - \overline{x}_i)^k (x_j - \overline{x}_j)^s \rho(t, x_t) dx_t, \quad (7)$$

$$\overline{x}_i = \int_{M_{t,u}} x_i \rho(t, x_t) dx_t, \quad i, j = 1, \ldots n, \quad j \geq i, \quad (8)$$

or

$$I(u) = \max_{t \in TN \in [0T]} \max_{x \in M_{t,u}} \varphi(t, x, \rho(t, x)) \to \min_{u \in D}. \quad (9)$$

OPTIMIZATION OF RADIAL MATCHING SECTION FOR RFQ CHANNEL

Consider the radial matching problem in RFQ accelerator. Let ellipse matrices $G_x(t, \varphi_0)$ and $G_y(t, \varphi_0)$, depending on time t and phase φ_0 , describe beam dynamics in radial matching section. Suppose, that $t = 0$ corresponds to the entrance into the radial matching section, and $t = T$ corresponds to the entrance into the regular part of accelerator. Suppose also, that acceptances of the regular part of the accelerator are known, i.e. the following matrices are given:

$$G_x(T, \varphi_0) = G_{x,T}(\varphi_0), \; G_y(T, \varphi_0) = G_{y,T}(\varphi_0). \quad (10)$$

The optimization problem for the radial matching section is to find a function $a(\tau)$, i.e. law of the radius change along the matching sections, providing under the conditions (10) the maximum possible overlapping of families of ellipses at the entrance of the radial matching section [3, 6, 7].

METHOD OF SOLUTION

Let's consider the following functional which characterizes the quality of the matching section by mismatch of ellipses $G_x(0, \varphi_0)$ and $G_y(0, \varphi_0)$ with given ellipses B_x and B_y:

$$J(a) = \max_{\varphi_0} \lambda_x^{-1}(\varphi_0) + \max_{\varphi_0} \lambda_y^{-1}(\varphi_0), \quad (11)$$

where

$$\lambda_x^{-1}(\varphi_0) = \lambda^{-1}(G_x(0, \varphi_0), B_x), \quad (12)$$

$$\lambda_y^{-1}(\varphi_0) = \lambda^{-1}(G_y(0, \varphi_0), B_y). \quad (13)$$

Here $\lambda = \min(\lambda_1, \lambda_2)$ is a minimum eigenvalue of a cluster of quadratic forms generated by a pair of ellipses with the matrices G and B:

$$\chi(\lambda) = \det(G - \lambda B) = 0, \quad \chi(\lambda_1) = \chi(\lambda_2) = 0. \quad (14)$$

The value of the inverse minimum eigenvalue characterizes the degree of mismatch pairs of ellipses. In the case of fully identical ellipses, this value is equal to unity. So always $\lambda^{-1} \geq 1$.

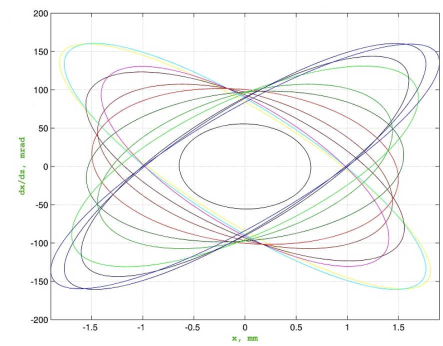

Figure 1: Acceptance without radial matching section, $(x, dx/dz, E_x = 0.050892\ \pi\cdot cm\cdot mrad)$

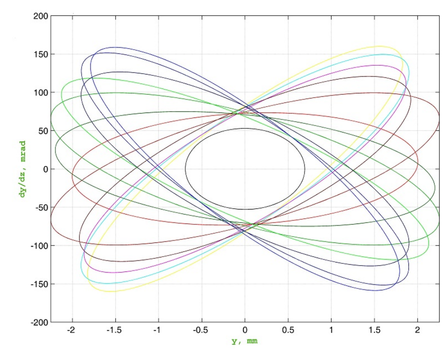

Figure 2: Acceptance without radial matching section $(y, dy/dz, E_y = 0.052122\ \pi\cdot cm\cdot mrad)$

Matrices B_x and B_y describing the desired phase portrait of the beam at the beginning of the matching section.

The problem of minimizing the functional (11) is the minimax optimization problem.

The analytical representation [6, 7] of the variation of the functional (11) were used to find geometric parameters of radial matching section of the RFQ accelerator of protons (initial energy 95keV, output energy 5 MeV, intervane voltage 100kV, RF field frequency 352 MHz, initial cell length 6.06 mm). Several of the possible choices of the law of variation of the channel radius along the radial matching section are presented in Fig. 5, 8, 11. In Fig. 1, 2 the RFQ acceptances without radial matching section are shown. The illustrations of effect of the radial matching sections (with channel radii presented at Fig. 5, 8, 11) are shown in Fig. 3-4, Fig. 6-7, Fig. 9-10 (correspondingly). The first variant (Fig. 3-5) is rather usual and requires converging ellipses at the entrance of the radial matching section. But the two others give us unusual results with neutral and divergent input ellipses: Fig. 6-8 and Fig. 9-11.

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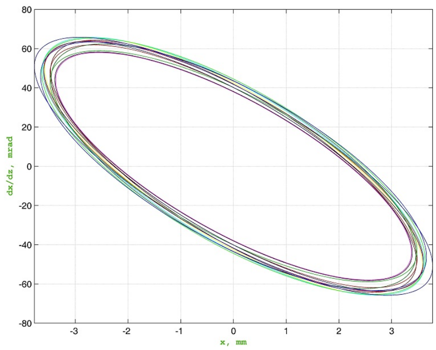

Figure 3: Acceptance with radial matching section ($x, dx/dz, E_x = 0.18393\ \pi\cdot cm\cdot mrad$)

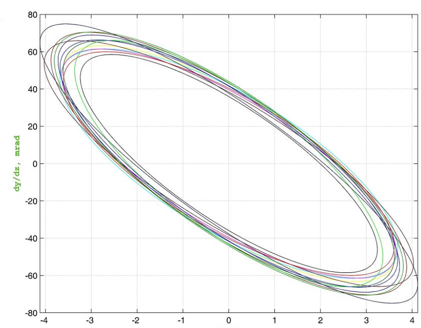

Figure 4: Acceptance with radial matching section ($y, dy/dz, E_y = 0.16571\ \pi\cdot cm\cdot mrad$)

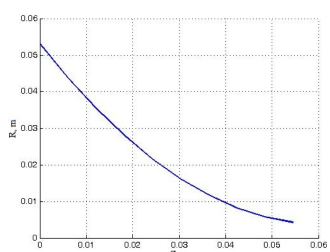

Figure 5: Radius of channel in radial matching section

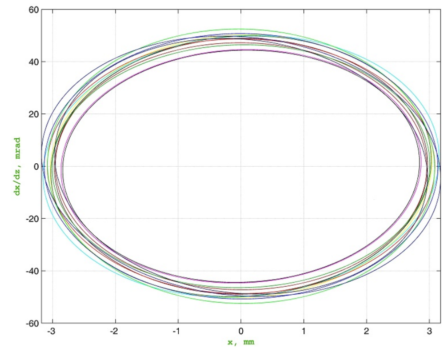

Figure 6: Acceptance with radial matching section ($x, dx/dz, E_x = 0.17945\ \pi\cdot cm\cdot mrad$)

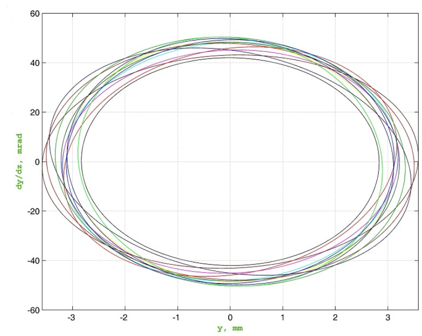

Figure 7: Acceptance with radial matching section ($y, dy/dz, E_y = 0.16955\ \pi\cdot cm\cdot mrad$)

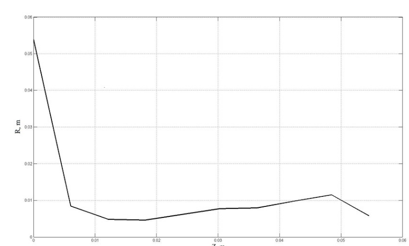

Figure 8: Radius of channel in radial matching section

SELF-CONSISTENT DISTRIBUTIONS

Consider nonrelativistic cylindrical beam in uniform longitudinal magnetic field. Assume that particles are distributed uniformly on the longitudinal coordinate z, and have the same longitudinal velocity. Such beam can be described by four-dimensional particle distribution in the phase space of the transverse motion. The well-known example is the Kapchinsky-Vladimirsky distribution [20].

Additionally assume that particles are evenly distributed on azimuthal angle φ and on phases of their trajectories θ. Then instead of four-dimensional distribution in the phase space, we can consider two-dimensional distribution in some two-dimensional cross-section of the phase space corresponding to the fixed values of φ and θ.

Let us take integrals of the transverse motion

$$M = r^2(\omega_0 + \dot{\varphi}), \quad H = \dot{r}^2 + \omega^2 r^2 + \frac{M^2}{r^2},$$

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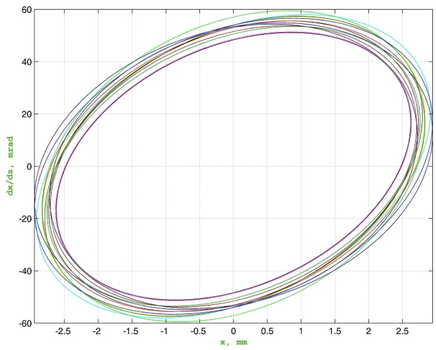

Figure 9: Acceptance with radial matching section ($x, dx/dz, E_x = 0.18023\ \pi\cdot cm\cdot mrad$)

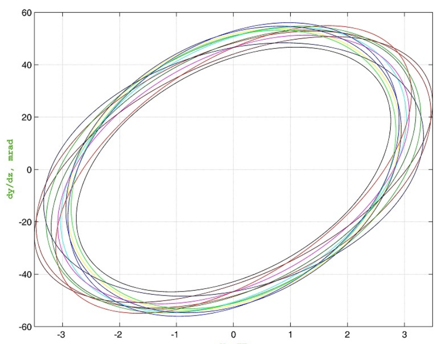

Figure 10: Acceptance with radial matching section ($y, dy/dz, E_y = 0.16834\ \pi\cdot cm\cdot mrad$)

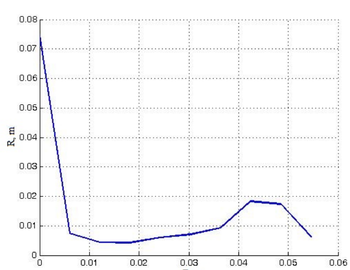

Figure 11: Radius of channel in radial matching section

as coordinates in this cross-section. By this reason, we will regard this cross-section as the space of the integrals of motion. Here $\omega^2 = \omega_0^2 - e\varrho_0/(m\varepsilon_0)$, $\omega_0 = eB_z/(2m)$, e and m are charge and mass of the particles, ϱ_0 is spatial density of the particles inside the beam cross-section, ε_0 is electric constant, B_z is longitudinal component of the magnetic flux density.

If beam is radially constrained, $r < R$, where R is beam radius, the possible values of M and H are bounded. For uniformly charged beam, the set Ω of permissible values M and H is described by inequalities [14–19]

$$2\omega|M| < H \leq M^2/R^2 + \omega^2 R^2, \qquad (15)$$

The set Ω is shown on Fig. 12.

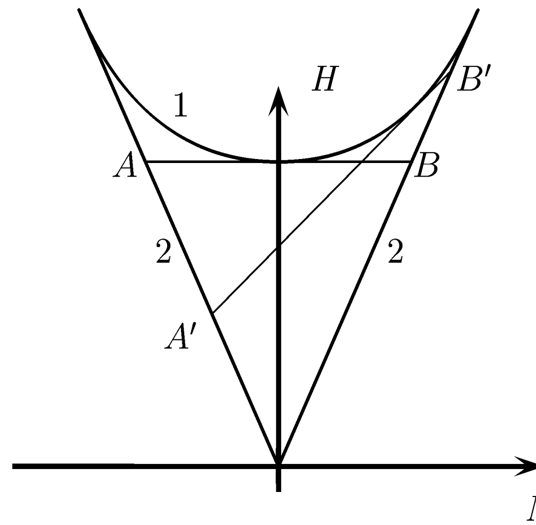

Figure 12: The set Ω for the uniform beam. The curve 1 represents the upper boundary of the set $\Omega : H = M^2/R^2 + \omega^2 R^2$. The segments 2 represent the lower boundary of the set $\Omega : H = 2\omega|M|$.

The simplest known distribution is the Brillouin flow [21]. For this distribution, all particles have the same values of integrals M and $H : M = 0, H = 0$.

If we specify particle density $f(M, H)$ in Ω, we get some self-consistent distribution.

When particles are uniformly distributed on segment AB, we get well-known Kapchinsky-Vladimirsky distribution. For this case, distribution density in the space of motion integrals is the simple layer which supporter is segment AB.

Another example is so called "rigid rotator distribution" [22]. In this case, particles are uniformly distributed on some segment $A'B'$ which is tangent to the upper boundary (15). KV distribution is a particular case corresponding to zero slope of segment $A'B'$. The mean angular momentum of the "rigid rotator distributions" is not equal to zero, except for the KV distribution. This can be seen from Fig. 12 where the parts of the segment $A'B'$ lying on the left and on the right of the axis H are not equal.

The beam is uniformly charged for all these distributions. By this reason, if we take any linear combination of these distributions, we get uniformly charged beam. It is easy to take such combination, that the mean angular momentum of particles will be equal to zero. Therefore such distribution can be taken as a

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model of a real beam, which is launched with the zero mean angular momentum.

Beam with slowly varied radius can be described analogously [17–19]. In this case, we take another integral of motion I instead of H. Integral I can be expressed through dynamical variables of a particle and the beam envelope. For this case, the set of permissible values of M and I looks like the set Ω on Fig. 12.

CONCLUSION

Mathematical models for beam dynamics optimization were presented. They may be applied to different dynamical systems.The optimization approach to finding geometric parameters of radial matching section is considered. It should be noted, that the proposed approach can be applied to optimize the transverse dynamics in accelerators if the dynamics is described by linear or nonlinear equations. In the case of nonlinear equations it is needed to consider RMS characteristics of the beam. In particular, this method can be used to minimize the growth of the effective emittance in the accelerators.

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