DEPENDENCE ON BETATRON OSCILLATIONS OF THE ANGULAR VELOCITY

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Abstract

Study of the topic, along with other factors necessary to understand the role of electron oscillations in the formation of the synchrotron radiation. The above-mentioned problem is considered by the author in various periodic magnetic fields. Emergence of the correction terms to the angular velocity of storage rings is also discussed.

At first let us take an axially-symmetric magnetic field as the most extensively studied. In the neighborhood of equilibrium orbit the magnetic field is chosen as

$$H_x = H_y = 0, H_z = br^{-n},$$

where b is the constant, $r = \sqrt{x^2+y^2}$, n is the field gradient ($0 < n < 1$). If $rot\bar{H} = 0$ one can concede that the potentials in cylindrical coordinates take the form

$$\Phi = 0, \quad A_r = A_z = 0,$$

$$A_\varphi = \frac{b}{r^{n-1}(2-n)}[1 + \frac{n(2-n)}{2r^2}z^2],$$

where φ is the azimuth angle. The relevant Hamiltonian $\mathcal{H}$ for electron can be written as

$$\sqrt{m^2c^4 + c^2p_z^2 + c^2p_r^2 + \frac{1}{r^2}c^2p_\varphi^2 + e^2A_\varphi^2 + 2ec\frac{A_\varphi}{r}p_\varphi}.$$

One of the Hamilton equations is

$$\dot{\varphi} = \frac{\partial\mathcal{H}}{\partial p_\varphi} = \frac{1}{\mathcal{H}}(\frac{c^2}{r^2}p_\varphi + \frac{ec}{r}A_\varphi),$$

where p_φ is the integral of motion.

Let us denote a small variable by $\rho = r - R$, where the equilibrium radius R can be deduced on condition that linear in ρ terms for Hamiltonian are absent (parabolic approximation). This gives

$$R = (\frac{2-n}{1-n}\frac{cp_\varphi}{eb})^{\frac{1}{2-n}}.$$

Restricting our selves to terms $\rho^2/R^2, z^2/R^2$, we can obtain the angular velocity as

$$\dot{\varphi} = \omega_0(1 - \frac{\rho}{R} + \frac{3-n}{2}\frac{\rho^2}{R^2} + \frac{n}{2}\frac{z^2}{R^2}), \qquad (1)$$

where $\omega_0 = ceH_0/E$ is the frequency, energy E is the constant, $H_0 = b/R^n$. The well-known asymptotics for oscillations have the form

$$\rho = A\cos(\sqrt{1-n}\omega_0 t + \chi), z = B\cos(\sqrt{n}\omega_0 t + \psi), \quad (2)$$

where A and$B$ are, respectively, the amplitudes of radial and axial vibrations, χ and ψ are the initial phases. Total velocity $v = \beta c$ is also constant and

$$v = R\omega_0\sqrt{1 + (1-n)\frac{A^2}{R^2} + n\frac{B^2}{R^2}}.$$

Expressions (1) and (2) made it possible to solve the synchrotron radiation problem [1] for given magnetic field. In such a case it has been found an essential influence of vertical oscillations on the spectral and angular properties of radiation in agreement with experiment. Here we can take only the linear terms and the accuracy was limited by decision (2).

Taking into account for oscillations the quadratic terms one can obtain nonlinear equations

$$\ddot{\rho} + (1-n)\omega_0^2\rho = \frac{\omega_0^2}{2R}[(1-n)(3+n)\rho^2 + n(1+n)z^2],$$

$$\ddot{z} + n\omega_0^2 z = n(1+n)\omega_0^2 z\frac{\rho}{R}.$$

Resolving them by the iterated method we can find, for example, an expression for radiation intensity [2] in a given approximation

$$W = W_0[(1 - \frac{n^2}{2})\frac{A^2}{R^2} - n\frac{3-2n+n^2}{2(1-n)}\frac{B^2}{R^2}],$$

where intensity for homogeneous magnetic field

$$W_0 = \frac{2}{3}\frac{ce^2}{R^2}\frac{\beta^4}{(1-\beta^2)^2}.$$

Clearly formula (1) may be also derived from equation

$$\frac{d}{dt}(r^2\dot{\varphi}) = -\frac{e_0}{mc}r(\dot{z}H_r - \dot{\rho}H_z),$$

where the constant of integration is defined as $\omega_0 R^2$.

In studies of radiation in the straight section accelerators we expanded in a power series the transverse components of magnetic field or gradient. Here an electron revolves on orbit consisting of N periods, where one element of the system includes a bending magnet of length $a = 2\pi R/N$ and free gap of length l. The length of entire orbit will be

$$2\pi R + Nl = 2\pi R_0,$$

where $R_0 = (1+k)R$ is the averaged radius, $k = l/a$. After expansion in a series we can put $n(\tau) = f(\tau)n$, where $\tau = N\varphi$ and the discrete function

$$f(\tau) = \frac{1}{1+k}[1 + \frac{2(1+k)}{\pi}\sum_{\nu=1}^{\infty}\frac{\sin\nu\tau_1}{\nu}\cos\nu(\tau-\tau_1)]$$

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with $\tau_1 = \pi a/(a+l)$. This operator $f(\tau)$ is 1 on the section with the magnetic field and 0 in the free gap.

The equation for the vertical oscillations of electron in the linear approximation can be put into the form

$$\frac{d^2 z}{d\tau^2} + \frac{(1+k)^2}{N^2} n f(\tau) z = 0. \quad (3)$$

Solution of Eq. (3) may be sought as a series [3], where n/N^2 is the small parameter. In the long run it will be a superposition of sinusoids and cosine curves with modulated amplitudes.

For the radial vibrations one can substitute in Eq. (3) n by $1-n$. These radial oscillations are small compared with the radius and have only a negligible impact on the radiation properties. In this connection it may be assumed that a particle moves along a circle with radius R_0, and the guiding magnetic field H can be averaged over the entire length of the period. Because of the magnetic field components take the form

$$H_z = H_0[\frac{1}{1+k} - \frac{\rho}{R} n f(\tau)], H_r = -H_0 \frac{z}{R} f(\tau), \quad (4)$$

where already $\rho = r - R_0$. The angular velocity of this motion can be represented by the expression

$$\dot{\varphi} = \frac{\omega_0}{1+k}[1 - \frac{\rho}{R_0} + \frac{3}{2}\frac{\rho^2}{R_0^2} + \frac{n}{R^2}\int (z\dot{z} - \rho\dot{\rho}) f(\tau) dt]. \quad (5)$$

Coefficient $1/(1+k)$ here is due to the straight sections.

For the FODO model $(n > 1)$ the focusing and defocusing segments have length a and are separated by field free section of length l. Then the orbit length of N elements is

$$2\pi R + 2Nl = 2\pi R_0.$$

Expanding $n(\varphi)$ in a Fourier series, we get

$$n(\tau) = \frac{4n}{\pi}\sum_{\nu=0}^{\infty}\frac{\sin(2\nu+1)\tau_2}{2\nu+1}\cos(2\nu+1)(\tau - \tau_2),$$

where $\tau_2 = \pi a/(2(a+l))$. In this case one can use the Eq. (3) and suppose that the parameter n/N^2 is else small. The guiding magnetic field will be more properly as

$$H_z = H_0[\frac{a}{a+l} + \Phi_1(\tau)],$$

where

$$\Phi_1 = \frac{2}{\pi}\sum_{\nu=1}^{\infty}\frac{\sin 2\nu\tau_2}{\nu}\cos 2\nu(\tau - \tau_2).$$

Then in Eq. (5) we must add the correction

$$\frac{\omega_0}{1+k}\int \frac{\dot{\rho}}{R}(1 + \frac{\rho}{R})\Phi_1(\tau) dt.$$

For storage rings we shall restrict our consideration to the case of the triplet achromat lattice [4]. It has the defocusing quadrupole of length a_1, then focusing quadrupoles of a long and bending magnet of length d lie on each side. The total run of lattice is

$$L = a_1 + 2(a + d + l_1 + l_2 + l_3),$$

where l_i is the length of free shifts. Let us denote the magnetic field of dipole by $B_z = B$, the lens constant by g, the small radial vibrations by x instead of ρ. The transverse components of magnetic field are

$$H_z = \frac{2d}{L} B - \Phi_2(\tau) g x + H_{ad}, \quad H_r = -\Phi_2(\tau) g z,$$

where

$$\Phi_2(\tau) = \frac{2a - a_1}{L} + \frac{2}{\pi}\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu}\cdot$$

$$[2\sin\tau_3 a \cos\tau_3(2l_1 + a + a_1) - \sin\tau_3 a_1]\cos\nu\tau,$$

with $\tau_3 = \pi\nu/L$.

Besides the part of guiding field is

$$H_{ad} = \frac{4B}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{\nu}\sin\tau_3 d\cos\tau_3(d + 2l_3)\cos\nu\tau,$$

which disappears after averaging. Eq. (5) is complemented by expression

$$\omega_q \frac{2a - a_1}{2L}(\frac{z^2}{R_0^2} - \frac{x^2}{R_0^2})$$

and $n\omega_0 f(\tau)/((1+k)R^2)$ is replaced by $\omega_q\Phi_2(\tau)/R_0^2$, where $\omega_q = e_0 g R/(mc), k = (L - 2d)/(2d)$.

The vertical oscillations are described by linear equation

$$\frac{d^2 z}{d\tau^2} + \frac{C}{N^2}\Phi_2(\tau)\cdot z = 0, \quad (6)$$

where $C = (1+k)\omega_q/\omega_0$. Note that parameter $C/N^2 >> 1$ and the quest of solution is a challenging task. It should also be pointed out that the expressions (3) and (6) is the Hill equations.

The used procedures are first of all stipulated by a necessity to investigate the synchrotron radiation characteristics in accelerators. Formulas (1) and (5) illustrate that $\dot{\varphi}$ is different in the various points of particle trajectory. Thus it was established that the angular velocity of particle is defined by the structure of concrete magnetic system. Proposed approach can be developed further and move to nonlinear problems.

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