EXAMINATION OF CHARGED PARTICLE DYNAMICS THROUGH EMPLOYMENT OF THE FOURIER SERIES

O.E. Shishanin, Moscow State Industrial University, Moscow

Abstract

To explore a role of betatron oscillations in accelerators at the generation of synchrotron radiation it was necessary to have the continuous solutions for the Hill equation. In this case the transverse components of magnetic field were expanded in a Fourier series [1, 2]. This approach was usefully employed for the above-mentioned problem and later it has become evident that the same scheme is usable for storage rings. The procedure may even work for wigglers and undulators.

As an illustration let us consider three examples for axial betatron oscillations. The FOFDOD model can be treated as a strongfocusing system in accelerators. Here notations F and D mean focusing and defocusing magnetic fields, respectively, and symbol O is assigned to field-free section. The length L of single period successively consists of $a+l_1+a+a+l_2+a$, where a is the length of magnets, l_1 and l_2 are the extension of straight shifts. The path of orbit S is defined as

$$2\pi R + N(l_1+l_2) = 2\pi R_0,$$

where

$$R_0 = R(1+k),\ k=(l_1+l_2)/4a,\ a=\pi R/2N,$$

N is the number of periods. Magnetic field gradient is equal to n at

$$\varphi \in (0, aT) \cup ((a+l_1)T, (2a+l_1)T),$$

where φ is the azimuth angle and $T = 2\pi/(LN)$. If

$$\varphi \in ((2a+l_1)T, (3a+l_1)T) \cup ((3a+l_1+l_2)T, 2\pi/N),$$

gradient will be $-n$. This periodic step function may be written after expansion as

$$n(\tau) = \frac{8n}{\pi}\sum_{\nu=1}^{\infty} g_\nu \cos\nu(\tau-\tau_1),$$

where $\tau = N\varphi$,

$$\tau_1 = (2a+l_1)\pi/L,\ g_\nu = \sin^2(\pi\nu a/L)\cdot\sin(\nu\tau_1)/\nu.$$

In the middle of the first magnet $\varphi = a\pi/LN$ and after summing $n(\tau)=n$. If take one-half of the second free section $\varphi = (2a+l_1+l_2/2)T$, then $n(\tau)=0$. For the points with first-kind discontinuities the function $n(\tau)$ equals, by the Dirichlet theorem, $0, n/2, -n/2$. The field of single magnet H is given as br^{-n}, where b is the constant.

The equation of vertical oscillations in linear approximation takes the form

$$\frac{d^2z}{d\tau^2} + \frac{(1+k)^2}{N^2}n(\tau)z = 0. \tag{1}$$

Setting $z = \exp(i\gamma_z\tau)\varphi_z(\tau)$, instead of (1) other equation for function $\varphi_z(\tau)$ may be derived as

$$\frac{d^2\varphi_z}{d\tau^2} + 2i\gamma_z\frac{d\varphi_z}{d\tau} + [\frac{(1+k)^2}{N^2}n(\tau) - \gamma_z^2]\varphi_z = 0.$$

Putting

$$\varphi_z = \varphi_0 + \sum_{i=1}^{\infty}\varphi_i/N^i,\ \gamma_z = \sum_{i=1}^{\infty}\gamma_i/N^i$$

and exclusive of secular terms one can sequentially obtain: $\varphi_0 = b,\ \varphi_1 = b_1$ (b and b_1 are the constants);

$$\gamma_1 = 0,\ \varphi_2 = bN^2S_1,\ \varphi_3 = b_1\varphi_2/b,$$

$$\gamma_2 = \pi n\sqrt{1+k}/2\sqrt{3},$$

where

$$S_1 = \frac{8n(1+k)^2}{\pi N}\sum_{\nu=1}^{\infty}\frac{g_\nu}{\nu^2}\cos\nu(\tau-\tau_1).$$

Then frequency is formed as $\nu_z = \gamma_z N$ and phase lag equals to well-known quantity

$$\mu_z = \pi^2 n\sqrt{1+k}/N^2\sqrt{3}.$$

The first terms of asymptotics can be written as follows:

$$z = B[(1+S_1)\cdot\cos\tau_z + \nu_z S_2\cdot\sin\tau_z], \tag{2}$$

where $\tau_z = \nu_z\tau/N + \psi$,

$$S_2 = \frac{16n(1+k)^2}{\pi N^3}\sum_{\nu=1}^{\infty}\frac{g_\nu}{\nu^3}\sin\nu(\tau-\tau_1).$$

Here B and ψ have been interpreted as the amplitude of axial oscillations and the initial phase. It is significant that the solution (2) is the superposition of harmonic curves with modulated amplitudes. The small parameter is n/N^2. To ease the task of estimation of an angular velocity $\dot\varphi$, the guiding magnetic field H_0 may be averaged over the entire period and R_0 can be conceived of as a mean radius. After that it is believed that $\dot\varphi$ becomes

$$\dot\varphi = \frac{\omega_0}{1+k}[1 - \frac{\rho}{R_0} + \frac{3}{2}\frac{\rho^2}{R_0^2} +$$

$$\int n(\tau)\left(\frac{z\dot{z}}{R^2} - \frac{\rho\dot{\rho}}{R^2}\right)dt],$$

where $\omega_0 = ceH/E,\ \rho = r - R_0$. The case discussed above was for $n(\tau)$ expansion.

By contrast, in the succeeding section let us consider a Fourier-series expansion of the magnetic field immediately for the storage ring.

In particular, Chasman-Green lattice [3, 4] has in centre focusing quadrupole. Next are magnets, defocusing and focusing quadrupoles by way of straight sections which lie on each side of the middle of period. For single lattice the length L equals $2d + 5a + 8l$, where d is the length of magnet, a is the length of quadrupoles, and l defines extension of free gaps. The path of orbit is

$$S = 2\pi R + N(5a + 8l),$$

parameter $k = (5a + 8l)/2d$.

The transverse coordinates are chose as x and z. As a consequence the quadrupole magnetic fields become

$$H_z^f = -gx,\ H_z^d = gx; \quad H_r^f = -gz,\ H_r^d = gz,$$

where g is the constant of lens, besides index f means focusing and d defocusing. The field of dipole B denotes B_z. The vertical component of magnetic field H_z after expansion is equal to

$$\frac{2d}{L}B - g\rho f(\tau) + B_a, \tag{3}$$

where

$$B_a = \frac{4B}{\pi}\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu}\sin\frac{\pi\nu}{L}d\cdot$$

$$\cos\frac{\pi\nu}{L}(a + d + 2l)\cdot\cos\nu\tau,$$

$$f(\tau) = \frac{a}{L} + \frac{2}{\pi}\sum_{\nu=1}^{\infty}\frac{f_\nu}{\nu}\cos\nu\tau$$

with

$$f_\nu = \sin\frac{\pi\nu}{L}a[(-1)^\nu +$$

$$4\sin\frac{\pi\nu}{L}(a + l)\cdot\sin\frac{\pi\nu}{L}(2a + 3l)].$$

After averaging of guiding magnetic field in (3) B_a is vanished.

The second component of field takes the form

$$H_r = -gzf(\tau).$$

An equation of axial oscillations becomes

$$\frac{d^2z}{d\tau^2} + \frac{C}{N^2}f(\tau)\cdot z = 0, \tag{4}$$

where $C = gR_0(1+k)/B$. In comparison with (1) there is constant term in last expression (4). Since sometimes a ratio $C/N^2 > 1$, asymptotics similar to (2) is unsuitable. Eq. (4) is the Hill equation with periodic coefficient and large parameter. Besides series

$$\sum_{\nu=1}^{\infty}(f_\nu/\nu)\cos\nu\tau$$

cannot be differentiated. Usual methods of calculation [5] here will not work. Nevertheless if take again the asymptotic in the same form one may obtain a frequency for this model

$$\nu_z = \sqrt{\frac{Ca}{L}}\left(1 + \frac{\pi Ca}{6N^2L}\right).$$

In this case the solution contains a set of sines and cosines with increasing amplitudes. Taking into account an injection of particles and marking off the certain points of trajectory one can enter the initial conditions and resolve the Cauchy problem for Eq. (4).

Finally let us call attention to undulators. The magnetic field strength in sinusoid case [6] is defined as

$$(0,\ 0,\ H\sin 2\pi x/l),$$

where l is the period of undulator. Here the longitudinal component of magnetic fields was neglected. Assume that the vertical part of field alternates up and down via gaps and has a rectangular shape. Using the Fourier expansion this field may be written as follows:

$$H_z = \frac{2H}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}[\sin(2k+1)\frac{2\pi}{l}x +$$

$$(-1)^k\cos(2k+1)\frac{2\pi}{l}x].$$

Apparently there is a need to examine the distinction between two relations. However it should be emphasized that in practice the distribution of magnetic field may not be right-angled. For the magnetic field of an arbitrary configuration one may use numerical methods of harmonic analysis.

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