VECTOR POTENTIAL AND STORED ENERGY OF A QUADRUPOLE MAGNET ARRAY

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Abstract

The vector potential, magnetic field and stored energy of a quadrupole magnet array are derived. Each magnet within the array is a current sheet with a current density proportional to the azimuthal angle 2θ and the longitudinal periodicity $\frac{(2m-1)\pi}{L}$. Individual quadrupoles within the array are oriented in a way that maximizes the field gradient. The array does not have to be of equal spacing and can be of a finite size, however when the array is equally spaced and is of infinite size the solution can be simplified. We note that whereas, in a single quadrupole magnet with a current density proportional to cos2θ the gradient is pure, such purity is not preserved in a quadrupole array.

1 INTRODUCTION

It has been proposed that commercial electricity can be generated economically from ion beam-driven fusion of deuterium and tritium in tiny target pellets [1]. A leading driver candidate is a high energy, high current heavy ion accelerator. To achieve high currents it is generally desirable to accelerate multiple beams in parallel through a low impedance accelerating structure; a long pulse induction linac can be designed to do this. Efficient transport of beam current in the multibeam accelerator would be accomplished with multiple channel superconducting quadrupole magnets operating in a DC mode with warm bore[2].

The vector potential and the magnetic field have been derived for an array of quadrupole magnets with a thin cos(2θ) current sheet at a radius r=R. [3][4]. The field strength within each coil varies purely as a Fourier sinusoidal series of the longitudinal coordinate z in proportion to ω_mz, where $\omega_m = \frac{(2m-1)\pi}{L}$, L denotes the *half-period*, and m is an integer associated with the longitudal harmonic. The analysis is based on the expansion of the vector potential in the region external to the windings of a single quad, and the use of the “Addition Theorem” to revise the expansion to one around any arbitrary point in space.

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The quad current density J (A/m) (a form that satisfies the conservation condition $\nabla \cdot \vec{J}_s = \frac{\partial J_z}{\partial z} + \frac{1}{R}\frac{\partial J_\theta}{\partial \theta} = 0$), is :

$$\vec{J}(\theta, z)|_{r=R} = \sum_{m=1} J_{0z,m} \left[\begin{array}{l} \left(\frac{\omega_m R}{2}\right) \sin 2\theta \sin \omega_m z \hat{e}_\theta + \\ \cos 2\theta \cos \omega_m z \hat{e}_z \end{array} \right]$$

$$J_{0z,m} = -\frac{1}{\mu_0} \frac{8 R G_{2,m}}{(\omega_m R)^3 K_2'(\omega_m R)}$$

$$\omega_m = \frac{(2m-1)\pi}{L}$$

$G_{2,m}$ is gradient at z=0 and L denotes the half period.

Quadrupoles are combined into an array with a center to center spacing of 2S and alternating current direction that maximizes the gradient (Fig. 1 and 2).

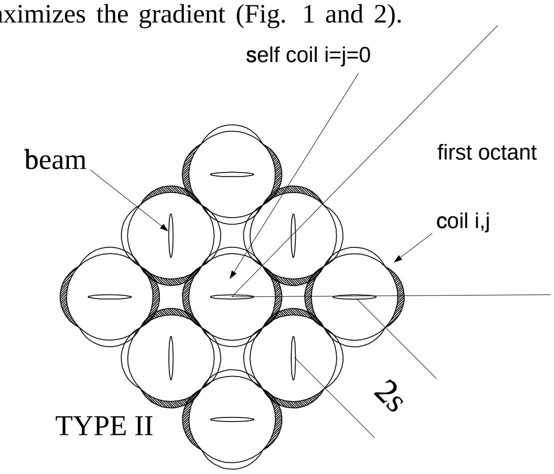

Figure 1: Cross section showing current density arrangement

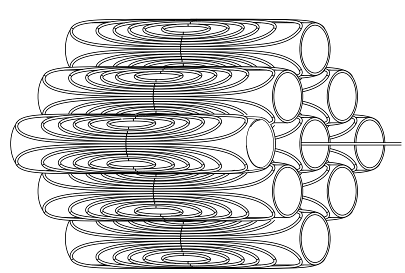

Figure 2: View of a 3x3 quadrupole array. The windings (of constant current) correspond to three terms m=1,2,3 which provide axial free space between arrays.

Based on such a current distribution the resulting vector-potential $\vec{A}$ and magnetic field $\vec{B}$ within the bore R of each quad, are:

$$\vec{A}_r = \begin{cases} \sum_{m=1} \sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2(\frac{\omega_m R}{2})^{2(2k-1)}} \cos 2(2k-1)\theta \\ \times \sin(\omega_m z) \begin{bmatrix} C^+_{k,m} I'_{2(2k-1)}(\omega_m r) + \\ 2(2k-1) C^-_{k,m} \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \end{bmatrix} \end{cases}$$

$$\vec{A}_\theta = -\begin{cases} \sum_{m=1} \sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2(\frac{\omega_m R}{2})^{2(2k-1)}} \sin 2(2k-1)\theta \\ \times \sin(\omega_m z) \begin{bmatrix} C^-_{k,m} I'_{2(2k-1)}(\omega_m r) + \\ 2(2k-1) C^+_{k,m} \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \end{bmatrix} \end{cases}$$

$$\vec{A}_z = \begin{cases} \sum_{m=1} \sum_{k=1} \frac{\mu_0 J_{0z,m} R (4k-3)!}{2(\frac{\omega_m R}{2})^{2(2k-1)}} \times \\ C^+_{k,m} I_{2(2k-1)}(\omega_m r) \cos 2(2k-1)\theta \cos \omega_m z \end{cases}$$

$$C^+_{k,m} = \sum_{i=1} \sum_{j=1} \begin{bmatrix} K_{4(k-1)}(\omega_m S_{i,j}) \cos 4(k-1)\theta_{0i,j} + \\ K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j} \end{bmatrix} \times \frac{16 I_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2}\right)^{4k-2}$$

$$C^-_{k,m} = \sum_{i=1} \sum_{j=1} \begin{bmatrix} K_{4(k-1)}(\omega_m S_{i,j}) \cos 4(k-1)\theta_{0i,j} - \\ K_{4k}(\omega_m S_{i,j}) \cos 4k\theta_{0i,j} \end{bmatrix} \times \frac{16 I'_2(\omega_m R)}{(4k-3)!} \left(\frac{\omega_m R}{2}\right)^{4k-1}$$

Where I_n and K_n are the "modified" Bessel functions of the first and second kind of order n, and the prime denotes differentiation with respect to the argument. The summation i,j is carried over the quads in the first octant of the array.

The magnetic field components are,

$$B_r = \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R}\right)^{4k-3} (4k-3)! \times C^-_{k,m} I'_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \cos \omega_m z$$

$$B_\theta = \sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R}\right)^{4k-3} 2(2k-1)(4k-3)! \times C^-_{k,m} \frac{I_{2(2k-1)}(\omega_m r)}{\omega_m r} \cos 2(2k-1)\theta \cos \omega_m z$$

$$B_z = -\sum_{m=1} \mu_0 J_{0z,m} \sum_{k=1} \left(\frac{2}{\omega_m R}\right)^{4k-3} (4k-3)! \times C^-_{k,m} I_{2(2k-1)}(\omega_m r) \sin 2(2k-1)\theta \sin \omega_m z$$

The format used here for $\vec{A}$ and $\vec{B}$ was specifically chosen to avoid a singularity that may rise when L is large (e.g. when the 3d problem reduces to 2d).

2 ANALYSIS

Consider a quadrupole with its center at $(S_{i,j} \cos\theta_0, S_{i,j} \sin\theta_0, z)$ as shown in Fig. 3. The expansion of the vector potential in the region outside the current sheet (ρ>R) and around that center is,

$$\vec{A}_\rho = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R) \cos 2\beta \sin \omega_m z \, [I_3(\omega_m R) K_3(\omega_m \rho) - I_1(\omega_m R) K_1(\omega_m \rho)]$$

$$\vec{A}_\beta = \sum_{m=1} \frac{\mu_0 J_{0z,m} R}{4} (\omega_m R) \sin 2\beta \sin \omega_m z \, [I_3(\omega_m R) K_3(\omega_m \rho) + I_1(\omega_m R) K_1(\omega_m \rho)]$$

$$\vec{A}_z = \sum_{m=1} \mu_0 J_{0z,m} R I_2(\omega_m R) K_2(\omega_m \rho) \cos 2\beta \cos \omega_m z$$

The relations between the components of the vector (A_r, A_θ, A_z) around (r, θ, z) and the above components (A_ρ, A_β, A_z) are,

$$A_r = A_\rho \cos(\theta - \beta) + A_\beta \sin(\theta - \beta)$$
$$A_\theta = -A_\rho \sin(\theta - \beta) + A_\beta \cos(\theta - \beta)$$
$$A_z = A_z$$

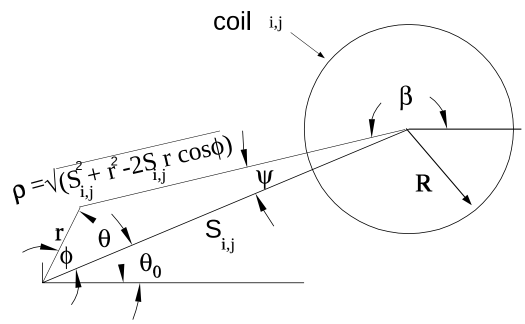

Figure 3: The geometry associated with the Addition Theorem[5].

If we wish to consider an ininite size array where the eight fold symmetry exists for each and every quad located at , $Se^{i\theta_0}$, we shall add the contributions of quads with their centers at $Se^{-i(\pm\theta_0)}$, $Se^{-i(\pm\frac{\pi}{2}\pm\theta_0)}$, $Se^{-i(-\pi\pm\theta_0)}$ and consider farther summations to be within the first octant only (quads on the symmetry line will require a weight factor of 1/2).

Once the vector potential has been derived, the field components within the bore R can then be calculated from,

$$\vec{B} = \nabla \times \vec{A}$$

Finally the arithmatic is check assuring the divergence of the vector potential and field are zero

$$\nabla \vec{A} = 0 \;\; , \;\; \nabla \vec{B} = 0$$

3 STORED ENERGY

The stored energy can be calculated by integrating the product of current density and vector potential$\vec{J} \cdot \vec{A}|_{r=R} = J_\theta A_\theta + J_z A_z$:

$$E = \frac{1}{2}\int\int\int \vec{J_a} \cdot \vec{A} dv = \frac{1}{2}\int_0^{2\pi}\int_{-L}^{L} \vec{J} \cdot \vec{A} R d\theta dz$$

(the current density is per unit length and the unit of energy is J).

Applying the orthogonality relations, the stored energy in a single quad is,

$$E_{total} = -\frac{\pi R^2 L \mu_0}{8} \sum_{m=1} \frac{J_{0z,m}^2 K_2^{'}(\omega_m R) I_2^{'}(\omega_m R) \times}{(\omega_m R)^2 \left[1 + \frac{4C_{1,m}^-}{(\omega_m R)^3 K_2^{'}(\omega_m R)}\right]}$$

where the second term in the square bracket corresponds to the contributions that arises from all neigboring coils in the array.

4 SIMULATION OF CURRENT DENSITY AND FLOW LINES

To generate flow lines we make use of a technique first demonstrated by J. Laslett and W. Fawley of this laboratory. The character of the flow lines (Figure 4) for a quadrupole magnet n=2 with a current density $\vec{J} = \sum_{m=1} J_{0z,m}\left[\left(\frac{\omega_m R}{2}\right)\sin 2\theta \sin \omega_m z \hat{e}_\theta + \cos 2\theta \cos \omega_m z \hat{e}_z\right]$ will follow by integrating the differential equation,$\frac{R d\theta}{dz} = \frac{J_\theta}{J_z}$, so that

$$\sin 2\theta = \frac{\sum_{m=1} J_{0z,m}}{\sum_{m=1} J_{0z,m}\cos \omega_m z}\sin 2\theta_0$$

where θ_0 denotes the value of θ at z=0.

In a special case, we may choose special values for $J_{02,m}$ such that,

$$J_{02,m} = J_{02}\frac{1}{2^{2(M-1)}}\frac{(2M-1)!}{(M+m-1)!(M-m)!}$$

where M corresponds to the number of m terms and J_{02} is a constant.

With that, the flow lines reduce to the simple expression,

$$\sin 2\theta = \frac{1}{\cos^{2M-1}\left(\frac{\pi z}{L}\right)}\sin 2\theta_0$$

and the current density components are,

$$\vec{J} = J_{02}\left\{\begin{array}{l} 0\hat{e}_r \\ \frac{\pi R}{2L}(2M-1)\cos^{2(M-1)}\frac{\pi z}{L}\sin\frac{\pi z}{L}\sin 2\theta\hat{e}_\theta \\ \cos^{2M-1}\frac{\pi z}{L}\cos 2\theta\hat{e}_z \end{array}\right\}$$

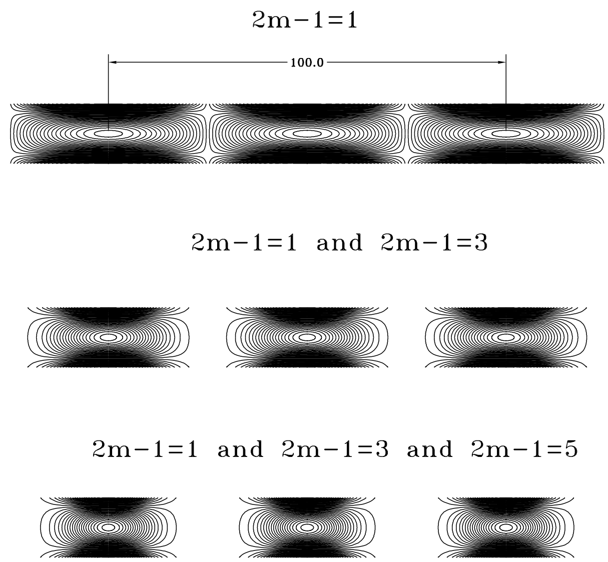

Figure 4: View of flow lines over a half period quad (M=1,2,3). These special cases reveal the reduction in crowding between magnets at the expense of an increased non-linear field.

5 CONCLUSION

The 3D expressions for the vector potential, field and energy have been derived. We note that neighboring coils within the array give rise to harmonic terms (m) which do not exist in a single quad with cos(2θ) current density. We also point out that the coefficients associated with $C_{k,m}^+$ in the vector potential drop out in the expressions for the field and energy.

6 REFERENCES

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