MULTITURN INJECTION OF AN ELECTRON BEAM IN A STORAGE RING

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Abstract

In report the radial-phase motion of electrons in a storage ring is investigated. The expression for Hamiltonian, connecting amplitudes of two modes of oscillations is received in view of nonlinear magnetic fields of the third order. Received Hamiltonian has a stationary point of a type "limiting cycle", which in coordinate space defines an equilibrium orbit. The existence "of a limiting cycle" allows to carry out multiturn injection of electron beam with final phase volume in a storage ring. The conditions of realization of multiturn injection are received.
The results of researches can be used and for stores of particles with large weight.

1 INTRODUCTION

Much attention (see, for example, [1-2]) were given to the study of a problem of a multiturn injection in storage rings. Researches of the particle injection in systems with stable motion round an equilibrium orbit is carried out in most cases. In the condition of a steady motion about closed orbit the "reflexive" Poincare theorem [3-4] is fulfilled which hampers a realization of multiturn injection of a charged particles beam.

In the this report the multiturn injection in a storage ring, in which there is no stable motion about closed orbit during injection, is investigated. In this case in the space of dynamic variables there can be a stationary point with the type of a "limiting cycle". A closed orbit becomes a limiting set for other trajectory. Particles can be injected in a storage ring along this trajectory unlimitedly long, and at small deviations from it - quite long. The "limit cycle" in a storage ring carry out with the help of impulse sextupole and octupole magnets. After realization of the injection the impulse elements switch off.

We obtained expressions for a Hamiltonian of a set of equations describing motion of electrons in a horizontal plane of a storage ring taking into account nonlinear magnetic fields. The task was solved in the first approximation by the Krylov-Bogolyubov method [5]. At deriving the short equations for guadrates of an oscillation amplitudes the representation of solutions in the Floquet form is used. The oscillations were not divided into fast "betatron" and slow "synchrotron" oscillations, and it allowed to define a Hamiltonian connecting amplitudes of these modes of oscillations.

2 THE EQUATIONS OF MOTION AND THEIR SOLUTION

The differential equations of motion of a charged particle in a horizontal plane of a storage ring look like [6]:

$$\frac{dx_l}{d\theta} - \sum_{m=1}^{4} A_{lm} x_m = F_l(x,\theta)$$

$$F_1(x,\theta) = -g x_2 x_3 + s x_2^2 - s x_2^2 x_3 + Q x_3^3 \qquad (1)$$

$$F_2(x_l,\theta) = F_3(x_l,\theta) = F_4(x_l,\theta) = 0\,,$$

where

$x_1 = dx_2/d\theta$,

x_2 - a deviation of an electron from equilibrium orbit,

x_3 - a relative deviation of an energy from an equilibrium value,

x_4 - a deviation of a phase of oscillations from an equilibrium value,

θ - azimuth coordinate.

Matrix ***(A)*** describe effect of electromagnetic fiels of storage ring in linear approximation without taken into account radiation $\boldsymbol{A(\theta+\theta_i) = A(\theta)}$,

$$g = -\frac{eK_0^{-2}}{p_0 c}\frac{dH_z}{dx_2}\,,\; s = -\frac{1}{2}\frac{eK_0^{-2}}{p_0 c}\frac{d^2 H_z}{dx_2^2}\,,$$

$$Q = -\frac{1}{6}\frac{eK_0^{-2}}{p_0 c}\frac{d^3 H_z}{dx_2^3}\,,$$

where

e - charge of an electron,

c - velocity of a light,

p_0 - impulse of an equilibrium orbit,

$K_0 = 2\pi/\boldsymbol{\Pi}$,

$\boldsymbol{\Pi}$ - perimeter of equilibrium orbit,

$\boldsymbol{\theta_i}$ – period,

H_z – axial component of a magnetic field.

In the equation (1) the most important nonlinear terms up to the 3d order inclusive are left. For the solution of this equation the method of average of Krylov-Bogolyubov [5] is applied, using representation of a solution of the homogeneous equation (1) ($F_1=0$) in the Floquet form:

$$x_l = C_1 f_l^{(1)} e^{i\psi_1\theta} + C_2 f_l^{(2)} e^{i\psi_2\theta} + c.c. \qquad (2)$$

where $f_l^{(1)}, f_l^{(2)}$ (l = ***1,2,3,4)*** periodic Floquet functions,

C_1, C_2 - constants.

In case of a piecewise constant dependence of the elements of a matrix *(A)* from ϑ, Floquet functions $f_l^{(1)}$, $f_l^{(2)}$ (l = ***1,2,3,4)*** and the magnitudes of Floquet indexes ψ_1, ψ_2 can be calculated.

Assuming that C_1 and C_2 are slowly varying functions of ϑ and substituting (2) in (1) and after the averaging it is possible to obtain a short differential equations for quadrates of an oscillation amplitude $I_1 = |C_1|^2$ and $I_2 = |C_2|^2$:

$$\frac{dI_1}{d\theta} = (L_1^s + L_1^Q)I_1^2 + (L_{12}^s + L_{12}^Q)I_1 I_2$$
$$\frac{dI_2}{d\theta} = (M_1^s + M_2^Q)I_2^2 + (M_{12}^s + M_{12}^Q)I_1 I_2 \tag{3}$$

where

$$L_1^s = \frac{1}{4}\left\langle\left\{s\Delta_1 f_{x_2}^{(1)}\left[2 f_{x_2}^{*(1)} f_{x_3}^{(1)} + f_{x_2}^{(1)} f_{x_3}^{*(1)}\right] + c.c.\right\}\right\rangle,$$

$$M_1^s = \frac{1}{4}\left\langle\left\{s\Delta_2 f_{x_2}^{(2)}\left[2 f_{x_2}^{*(2)} f_{x_3}^{(2)} + f_{x_2}^{(2)} f_{x_3}^{*(2)}\right] + c.c.\right\}\right\rangle,$$

$$L_{12}^s = \frac{1}{2}\left\langle\left\{s\Delta_1\left[\left|f_{x_2}^{*(2)}\right|^2 f_{x_3}^{(1)} + f_{x_2}^{(1)} f_{x_2}^{*(2)} f_{x_3}^{(2)} + f_{x_2}^{(1)} f_{x_2}^{(2)} f_{x_3}^{*(2)}\right] + c.c\right\}\right\rangle$$

,

$$M_{12}^s = \frac{1}{2}\left\langle\left\{s\Delta_2\left[\left|f_{x_2}^{*(1)}\right|^2 f_{x_3}^{(2)} + f_{x_2}^{*(1)} f_{x_2}^{(2)} f_{x_3}^{(1)} + f_{x_2}^{(1)} f_{x_2}^{(2)} f_{x_3}^{*(2)}\right], + c.c\right\}\right\rangle.$$

$$L_1^Q = -\frac{3}{4}\left\langle Q\left|f_{x_2}^{(1)}\right|^2\left(\Delta_1 f_{x_2}^{(1)} + c.c.\right)\right\rangle,$$

$$M_1^Q = -\frac{3}{4}\left\langle Q\left|f_{x_2}^{(2)}\right|^2\left(\Delta_2 f_{x_2}^{(2)} + c.c.\right)\right\rangle,$$

$$L_{12}^Q = -\frac{3}{2}\left\langle Q\left|f_{x_2}^{(2)}\right|^2\left(\Delta_1 f_{x_2}^{(1)} + c.c.\right)\right\rangle,$$

$$M_{12}^Q = -\frac{3}{2}\left\langle Q\left|f_{x_2}^{(1)}\right|^2\left(\Delta_2 f_{x_2}^{(1)} + c.c.\right)\right\rangle,$$

$\langle F(\theta)\rangle$ stands for $\lim\limits_{T\to\infty}\frac{1}{T}\int_0^T F(\tau)d\tau$.

At a normalization of the Floquet function the value of a wronskian Δ is put equal to "-4":

$$\Delta = \begin{vmatrix} f_{x_1}^{(1)} & f_{x_1}^{*(1)} & f_{x_1}^{(2)} & f_{x_1}^{*(2)} \\ f_{x_2}^{(1)} & f_{x_2}^{*(1)} & f_{x_2}^{(2)} & f_{x_2}^{*(2)} \\ f_{x_3}^{(1)} & f_{x_3}^{(1)} & f_{x_3}^{(2)} & f_{x_3}^{*(2)} \\ f_{x_4}^{(1)} & f_{x_4}^{*(1)} & f_{x_4}^{(2)} & f_{x_4}^{*(2)} \end{vmatrix} = -4 \tag{4}$$

Δ_1, Δ_2 - cofactors of a matrix Δ. Δ_1 - first column and first line, Δ_2- third column and first line. At an averaging it was supposed that $\psi_1 \neq \psi_2$, $\psi_1 \neq K$, $\psi_2 = K$, and also that none of the resonance conditions is fulfilled: $n\psi_1 + j\psi_2 \neq q, n, j, k, q = (0, \pm1, \pm2, ...)$.

It is possible to show, that:

$$\left(L_1^s + L_1^Q\right) = -\left(M_{12}^s + M_{12}^Q\right)/2;$$
$$\left(M_2^s + M_2^Q\right) = -\left(L_{12}^s + L_{12}^Q\right)/2 \tag{5}$$

In consequence the expression (5), the equations (3) has the following Hamiltonian:

$$H = I_2^2 I_1(M_2^s + M_2^Q) - I_1^2 I_2(L_1^s + L_1^Q) = I_2^2(0)I_1(0)(M_2^s + M_2^Q) - I_1^2(0)I_2(0)(L_1^s + L_1^Q) \tag{6}$$

where $I_1(0)$ and $I_2(0)$ - initial values of quadrates of amplitudes.

The connection between I_1 and I_2 is stipulated by an unresonance interaction of azimuth harmonics of sextupole and octupole components of a magnetic field of a storage ring with modulation of oscillation amplitudes of a particle in a periodic lattice of a storage ring.

$I_1(0)$ and $I_2(0)$ are expressed through initial values x_l and value of Floquet functions $f_l^{(1)}$, $f_l^{(2)}$ on an azimuth of injection by the following formulas:

$$I_1(0) = a_1^2 + b_1^2$$
$$a_1 = \frac{\sum_{l=1}^{4}\bar{x}_l\varphi_{l1}}{\Phi}, \quad b_1 = \frac{\sum_{l=1}^{4}\bar{x}_l\varphi_{l2}}{\Phi} \tag{7}$$

$$I_2(0) = a_2^2 + b_2^2$$
$$a_2 = \frac{\sum_{l=1}^{4}\bar{x}_l\varphi_{l3}}{\Phi}, \quad b_2 = \frac{\sum_{l=1}^{4}\bar{x}_l\varphi_{l4}}{\Phi} \tag{8}$$

where $\boldsymbol{\Phi}$ - determinant of a matrix $\boldsymbol{(\Phi)}$

$$(\Phi) = \begin{pmatrix} 2\,\mathrm{Re}\,\bar{f}_x^{(1)} & -2\,\mathrm{Im}\,\bar{f}_{x_1}^{(1)} & 2\,\mathrm{Re}\,\bar{f}_{x_1}^{(2)} & -2\,\mathrm{Im}\,\bar{f}_{x_1}^{(2)} \\ 2\,\mathrm{Re}\,\bar{f}_{x_2}^{(1)} & -2\,\mathrm{Im}\,\bar{f}_{x_2}^{(1)} & 2\,\mathrm{Re}\,\bar{f}_{x_2}^{(2)} & -2\,\mathrm{Im}\,\bar{f}_{x_2}^{(2)} \\ 2\,\mathrm{Re}\,\bar{f}_{x_3}^{(1)} & -2\,\mathrm{Im}\,\bar{f}_{x_3}^{(1)} & 2\,\mathrm{Re}\,\bar{f}_{x_3}^{(2)} & -2\,\mathrm{Im}\,\bar{f}_{x_3}^{(2)} \\ 2\,\mathrm{Re}\,\bar{f}_{x_4}^{(1)} & -2\,\mathrm{Im}\,\bar{f}_{x_4}^{(1)} & 2\,\mathrm{Re}\,\bar{f}_{x_4}^{(2)} & -2\,\mathrm{Im}\,\bar{f}_{x_4}^{(2)} \end{pmatrix},$$

φ_{ll} - cofactors of 1st column of a matrix $\boldsymbol{(\Phi)}$.

The line above $\bar{x}_l$, $\bar{f}_{x_l}^{(1)}$ and $\bar{f}_{x_l}^{(2)}$ stands for initial values of coordinates and Floquet functions on an azimuth of injection.

At $H=0$ equations (6) have two solutions:

1. $I_1(0) = I_2(0) = 0$ and according (3) $\left(\frac{dI_1}{d\theta}\right)_0 = 0$ and $\left(\frac{dI_2}{d\theta}\right)_0 = 0$. The point in a phase space, which is taken by a particle moving about equilibrium orbit corresponds to this solution.
2. $I_1(0) \neq 0, I_2(0) = 0$;

$$I_2(0) = I_1(0)\frac{\left(L_1^s + L_1^Q\right)}{\left(M_2^s + M_2^q\right)} \tag{9}$$

If the condition (9) is fulfilled, and further at a modification of ϑ will be fulfilled the following relation:

$$I_2 = I_1 \frac{\left(L_1^s + L_1^Q\right)}{\left(M_2^s + M_2^q\right)} \tag{10}$$

Substituting (5) and (10) in (3) we shall receive:

$$\frac{dI_1}{d\theta} = -\left(L_1^s + L_1^Q\right) I_1^2 \tag{11a}$$

$$\frac{dI_2}{d\theta} = -\left(M_1^s + M_1^Q\right) I_2^2 \tag{11b}$$

Integrating (11a) and (11b) we shall receive:

$$I_1 = \frac{1}{1/I_1(\theta) + \left(L_1^s + L_1^Q\right)\theta}; \quad I_2 = \frac{1}{1/I_2(\theta) - \left(M_2^s + M_2^Q\right)\theta} \tag{12}$$

It is made out from the relation (6) that for $H=0$ it is necessary that the magnitudes $(L_1^S + L_1^Q)$ and $(M_2^S + M_2^Q)$ were should be with identical signs. And it is clear from (12) that for I_1 and I_2 being aimed to zero at $\theta \to \infty$ it is necessary that the conditions $(L_1^S + L_1^Q) > 0$ and $(M_2^S + M_2^Q) > 0$ were fulfilled.

On account of relations (12) and (11a) and (11b) at $(L_1^S + L_1^Q) > 0$ and $(M_2^S + M_2^Q) > 0$ at $\theta \to \infty$ the particles, initial conditions of which satisfy the condition (10), will be indefinitely long coming close to a fixed point with parameters $\left(\frac{dI_1}{d\theta}\right) = 0$, $\left(\frac{dI_2}{d\theta}\right) = 0$, $I_1 = I_2 = 0$, that is to an equilibrium orbit. Thus, for a realisation of multiturn injection the realisation of two conditions $(L_1^S + L_1^Q) > 0$, $(M_2^S + M_2^Q) > 0$ and $H=0$ is necessary and sufficient.

In common case:

$$I_2 = \frac{1}{2}\frac{L_1}{M_2} I_1 \pm \sqrt{\frac{1}{4}\frac{I_1^2 L_1^2}{M_2^2} + \frac{H}{I_1 M_2}}, \tag{13}$$

Where I_1 is a solution of the differential equation:

$$\frac{dI_1}{\pm 2(M_2^s + M_2^Q)\sqrt{\frac{1}{4}\frac{I_1^2 L_1^2}{M_2^2} + \frac{H}{I_1 M_2}}} = d\theta. \tag{14}$$

For short here and below it is necessary: $(L_1^S + L_1^Q) = L_1$ e $(M_2^S + M_2^Q) = M_2$.

Assuming $H = -\overline{H}, \overline{H} > 0, M_2 > 0$ and by integrate (14) we shall receive expressions:

$$-\frac{2}{3}\frac{1}{L_1}\ln\frac{1 - \sqrt{1 - \left(I_{1\min}/I_1(0)\right)^3}}{1 + \sqrt{1 - \left(I_{1\min}/I_1(0)\right)^3}} = \theta, \tag{15}$$

where $I_{1\min} = \left(\frac{4\overline{H}M_2}{H_1^2}\right)^{1/3}$ - minimum value, which will take I_1 during the injection. At $I_1 = I_{1min}$ $dI_1/d\theta = 0$ and after reaching this value the increase of I_1 will begin.

From (15) follows, that number of turns during which it is possible to carry out the injection is equal to:

$$n = -\frac{2}{3\pi}\frac{1}{L_1}\ln\frac{1 - \sqrt{1 - \left(I_{1\min}/I_1(0)\right)^3}}{1 + \sqrt{1 - \left(I_{1\min}/I_1(0)\right)^3}}. \tag{16}$$

Assuming $I_{1min}/I_1(0) \ll 1$ we shall receive the approximate formula for an estimation of number of turns - n.

$$n = -\frac{2}{3\pi}\frac{1}{L_1}\ln\left(\frac{\left(I_{1\min}/I_1(0)\right)^3}{4}\right). \tag{17}$$

From (17) follows, that less is L_1, the more turns it is possible to inject, but L_1 should be great enough to ensure the separation of a beam from a septum for one turn. Integrating (14) and assuming $\theta = 2\pi$ we shall receive:

$$L_1 = \frac{1}{\pi}\frac{I_1(0) - I_1(2\pi)}{I_1(0)}. \tag{18}$$

Substituting L_1 from (18) to (17) we shall receive:

$$n = -\frac{2}{3}\left(\frac{I_1(0)}{I_1(0) - I_1(2\pi)}\right)\ln\left(\frac{\left(I_{1\min}/I_1(0)\right)^3}{4}\right). \tag{19}$$

Magnitude of a phase volume of a beam in space of x_i, which can be injected in a storage ring during n turns (17) is equal to the volume limited by the surface, the parameters of which can be obtained at a substitution in the equation (6) expressions for $I_1(0)$ and $I_2(0)$ from (7) and (8).

3 CONCLUSION

The method of an average applied in the work for a research of radial-phase motion of electrons in a storage ring, is approximate. The obtained formulas will be describe dynamics of particles well at realization of the condition $2\pi/\theta_i \gg 1$.

For a realization of conditions for existence of "limit cycle" it is necessary to select a storage system with close values ψ_1 and ψ_2. In common case sextupole and octupole fields result to the mutual transmission of an energy of synchrotron and betatron oscillations.

Multiturn injection can be used for the injection of heavy particles.

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