HYBRID COMPUTATION OF NORMAL MODE TUNE SHIFTS IN ROUNDED-RECTANGULAR PIPES*

V. Galdi, D.I.I.I.E. - Univ. of Salerno, I-84084 Fisciano (SA), Italy
S. Petracca†‡, and I. M. Pinto, Univ. of Sannio at Benevento, I-82100 Benevento, Italy

Abstract

A fast and accurate hybrid (analytical-numerical) technique for computing the normal mode tune-shifts in rounded-rectangular (stadium) pipes is introduced based on Galerkin method together with a smart representation of Poisson's equation Green's function in a rectangular domain. Comparison with standard finite-elements and finite difference methods shows that our method is faster and more accurate, requiring no numerical differentiation.

1 THE PROBLEM

Many beam-pipe cross-section geometries of potential interest for accelerators, including the stadium-shaped one recently proposed for LHC [3], differ from the rectangle only by the rounding of corners, or the substitution of straight sides with circular arcs. Computing the related betatron tune-shifts, resulting from collective (space-charge and image) effects is a key problem to prevent resonant betatron excitations leading to potentially harmful beam instabilities. The normal mode coherent and incoherent[1] tune-shifts can be written in terms of the normal mode Laslett coefficients ϵ as follows [2]:

$$\Delta\nu = -\frac{NRr_0}{\pi\nu\beta_0^2\gamma_0 L^2}\,\epsilon, \tag{1}$$

where[2]:

$$\epsilon_{1,2} = \frac{L^2}{4\Lambda}\left\{-\frac{\delta_x\partial_x\Phi^{(im)} + \delta_y\partial_y\Phi^{(im)}}{2} + \right.$$
$$\pm\left[\left(\frac{\delta_y\partial_y\Phi^{(im)} - \delta_x\partial_x\Phi^{(im)}}{2}\right)^2 + \right.$$
$$\left.\left. + \delta_x\partial_x\Phi^{(im)}\delta_y\partial_y\Phi^{(im)}\right]^{1/2}\right\}, \tag{2}$$

$\Phi^{(im)}$ is the image-potential produced in the beam pipe by a linear charge density Λ going through the beam center of charge $\underline{r}_b$, N is the number of particles in the beam, R is the machine radius, r_0 is the classical particle radius, L is a scaling length (usually, the maximum pipe diameter), ν is the nominal tune, and

$$\begin{cases} \delta_{x,y} = \partial_{x,y}|_{\underline{r}=\underline{r}_b}, \text{ incoherent regime,} \\ \delta_{x,y} = (\partial_{x,y} + \partial_{x_b,y_b})|_{\underline{r}=\underline{r}_b}, \text{ coherent regime.} \end{cases} \tag{3}$$

2 THE METHOD

For computing the image potential $\Phi^{(im)}$ in rounded rectangular geometries, it is convenient to use the rectangular-domain Green's function g_R (henceforth RDGF), viz. :

$$\Phi^{(im)}(\underline{r},\underline{r}_b) = \Phi(\underline{r},\underline{r}_b) - \Lambda g_0(\underline{r},\underline{r}_b),$$
$$\Phi(\underline{r},\underline{r}_b) = \Lambda\left[\sum_k \int_{\sigma_k} g_R(\underline{r},l_k)\rho_{\sigma_k}(l_k)dl_k + g_R(\underline{r},\underline{r}_b)\right], \tag{4}$$

where g_0 is the free-space Green's function, the unknown ρ_{σ_l} are obviously nonzero *only* on the rounded portion of ∂S_0, i.e., the arcs σ_k, and l_k is the arc-length on σ_k.

We seek a hybrid (analytical-numerical) solution of eq. (4) by using Galerkin (moments) method [6], whereby we first expand the unknown ρ_s:

$$\rho_{\sigma_k}(l_k) = \sum_{n=1}^{N} b_n^{(k)} w_n^{(k)}(l_k), \tag{5}$$

into a suitable (finite) set of basis functions $\{w_1^{(k)}(l_k),\ldots,w_N^{(k)}(l_k)\}$, defined on σ_k, where $\{b_1^{(k)},...,b_N^{(k)}\}$, are N-dimensional vectors of unknown coefficients, and then enforce the (Dirichlet) boundary conditions on the arcs σ_k, whence:

$$\int_{\sigma_k} \Phi(l_k,\underline{r}_b) w_n^{(k)}(l_k)dl_k = 0,$$
$$n = 1,2,\ldots,N;\ k = 1,2,\ldots,P, \tag{6}$$

thus obtaining a block-matrix linear system:

$$[\mathbf{L}]\ \mathbf{b} = \mathbf{c}. \tag{7}$$

The matrix $[\mathbf{L}]$ is readily shown to be symmetrical, positive definite and hence non-singular. The components of $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{L}$ are explicitly given by (5),

$$c_i^{(k)} = -\int_{\sigma_k} w_i^{(k)}(l_k) g(l_k,\underline{r}_b)dl_k,$$

* Work supported in part by INFN.

† On leave of absence from KEK, Tsukuba, JPN.
‡ e-mail stefania@kekvax.ac.jp

[1] The *incoherent* and *coherent* regimes correspond to $\underline{r} \neq \underline{r}_b = \underline{r}_{eq}$ and $\underline{r} = \underline{r}_b \neq \underline{r}_{eq}$, respectively, $\underline{r}_{eq}$ denoting the beam center of charge equilibrium position [2].

[2] The pipe-shape independent space-charge contribution to the tune-shift is neglected here for simplicity.

$$i = 1, 2, \ldots, N; \;\; k = 1, 2, \ldots, P, \tag{8}$$

and:

$$[L_M^{(p,q)}]_{ij} = \int_{\sigma_p}\int_{\sigma_q} g(l_p, l_q) w_i^{(p)}(l_p) w_j^{(q)}(l_q)\, dl_p\, dl_q,$$

$$i, j = 1, 2, \ldots, N, \;\; p, q = 1, 2, \ldots, P, \tag{9}$$

where the upper indexes identify the block sub-matrix, and the lower ones the element in each sub-matrix.

Using eq.s (2)-(5), once (7) has been solved, the Laslett coefficients can be computed *without* resorting to numerical differentiation. This makes the proposed method definitely more accurate than both finite-differences and finite-elements.

3 IMPLEMENTATION AND COMPUTATIONAL BUDGET

Fast and accurate numerical solution of (7) follows from a skillful choice of the RDGF representation in (4) and the basis functions in (5).

A rapidly converging series expansion of the RDGF [5] , which explicitly contains the (logarithmic) singular term is [3]:

$$g_R(\underline{r}, \underline{r}_b) = -\sum_{m=-\infty}^{\infty} \log \frac{T_m^{10}(\underline{r}, \underline{r}_b) T_m^{01}(\underline{r}, \underline{r}_b)}{T_m^{00}(\underline{r}, \underline{r}_b) T_m^{11}(\underline{r}, \underline{r}_b)}, \tag{10}$$

where:

$$T_m^{pq}(\underline{r}, \underline{r}_b) = 1 + \exp\left[-2\,|y - (-)^p y_b + 2bm|\,\frac{\pi}{a}\right] +$$

$$-2\exp\left[|y - (-)^p y_b + 2bm|\frac{\pi}{a}\right]\cos\left[\frac{\pi}{a}\,(x - (-)^q x_b)\right], \tag{11}$$

a, b being the rectangle side lengths.

A convenient set of (partially overlapping) piece-wise parabolic *subdomain* basis functions, can be defined in terms of the local angles ϕ (we drop the suffix k for simplicity) as follows:

$$w_i(\phi) = \frac{\Delta\phi^2 - (\phi - \phi_i)^2}{\Delta\phi^2},$$

$$\phi_i - \Delta\phi(1 - \delta_{i1}) \le \phi < \phi_i + \Delta\phi(1 - \delta_{iN}),$$

$$i = 1, 2, \ldots, N, \tag{12}$$

where $\Delta\phi$ is the angular discretization step (assumed the same for all arcs), ϕ is related to the local arc-length l by $l = R\phi$, R being the local curvature radius, and δ_{rs} is the Kronecker symbol[4]. The relevant local coordinate systems are sketched in Fig. 1. Note that: *i)* the choice of *subdomain* basis functions, rather than *full-domain* ones, results into *fewer* singular integrals in $[\mathbf{L}]$; *ii) no* polygonal approximation of the arcs is implied, resulting into fewer functions being needed for a given accuracy.

Letting P the number of arcs in the rounded portion of ∂S_0, the system (7) has rank NP. Computing the matrix elements requires evaluating up to $PN(PN-1)/2$ double-integrals[5]. These latter can be either evaluated numerically using standard routines appropriate for regular [7] and singular integrands [8], or analytically [4]. Matrix inversion for solving (7) is not the most demanding task, in view of the typically small ($NP \approx 20$) $\mathbf{L}$ matrix size. In all numerical simulations below we truncated (10) at $|m| \le 3$ and took $\Delta\phi = \pi/10$, corresponding to a matrix size $NP = 20$.

4 NUMERICAL RESULTS AND CONCLUSIONS

The circular pipe, for which the tune-shifts are known exactly, is the hardest conceivable test case for the proposed method (largest departure from rectangular geometry). It is seen from Fig. 2 that the obtained accuracy is very good.

Our method was subsequently applied [4] to a number of different proposed geometries relevant to LHC [3].

As an example the contour-level plots for the incoherent and coherent (both normal modes) Laslett coefficients for a stadium-shaped pipe, sketched in Fig. 3, are shown in Fig.s 4-6.

As a conclusion, we found that the above hybrid approach is comparatively faster and more accurate than available finite-element and/or finite-difference techniques.

5 REFERENCES

[1] L.J. Laslett, Proc. of the 1963 Summer Study BNL 7534.
[2] S. Petracca, Particle Accel., **48**, 81, 1994.
[3] The LHC study group, CERN/AC/95-05 (LHC), 1995.
[4] V. Galdi, S. Petracca, I. Pinto, Particle Accel., PA(R)-85, 1999, in print.
[5] M. Bressan and G. Conciauro, Alta Frequenza, **LII**, 188, 1983, ibidem **LVII**, 217, 1988.
[6] R.F. Harrington, *Field Computation by Moment Methods*, McMillan, New York, 1961.
[7] W.H. Press et al., *Numerical Recipes, the Art of Scientific Computing*, 2nd Ed., Cambridge Un. Press, 1992.
[8] SLATEC Pub. Dom. Math. Lib., NETLIB, *http://www.netlib.org*

[3] It is easily recognized that the (logarithmic) singularity of g_R appears in the T_0^{10} term.

[4] For $i = 1, N$, eq. (12) yields the correct behaviour at the points where the circular arcs join the straight portions of ∂S_0, where ρ_s can be different from zero, but its derivative should vanish.

[5] Due to geometrical (specular) symmetries, the effective number of elements to compute is usually smaller.

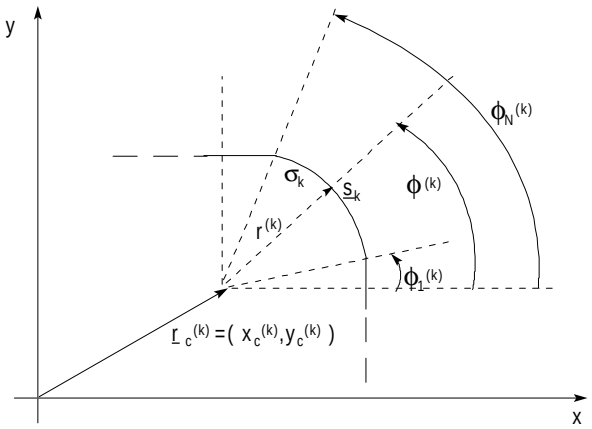

Fig. 1 - Local coordinate system relevant to eq. (12).

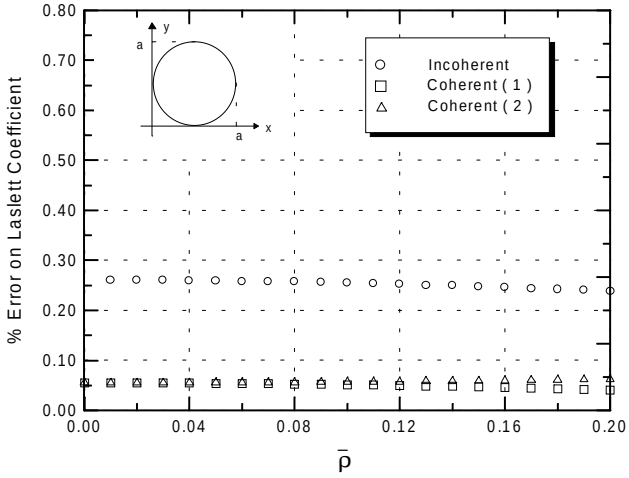

Fig. 2 - Circular pipe. Errors on Laslett coefficients vs. scaled radial distance, $\bar{\rho} = \frac{2}{a}\left[(x-a/2)^2 + (y-a/2)^2\right]^{1/2}$.

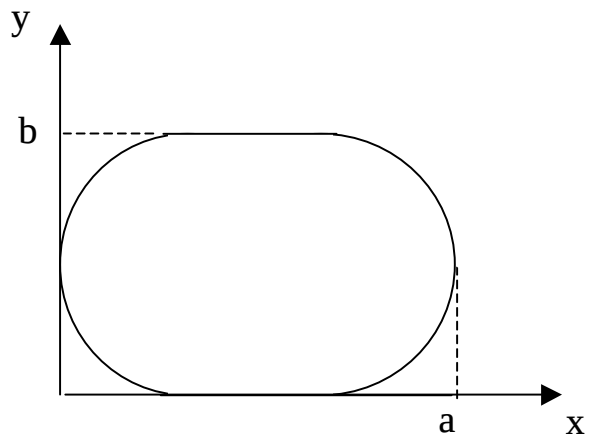

Fig. 3 - Stadium-shaped pipe (a=1, b=0.7).

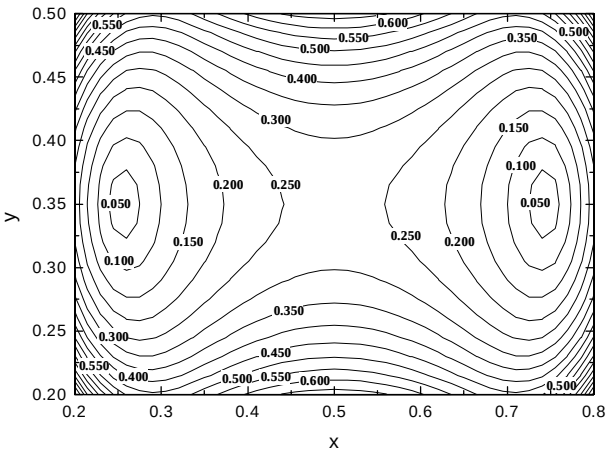

Fig. 4 - Stadium-shaped pipe. Incoherent Laslett coefficients (both normal modes).

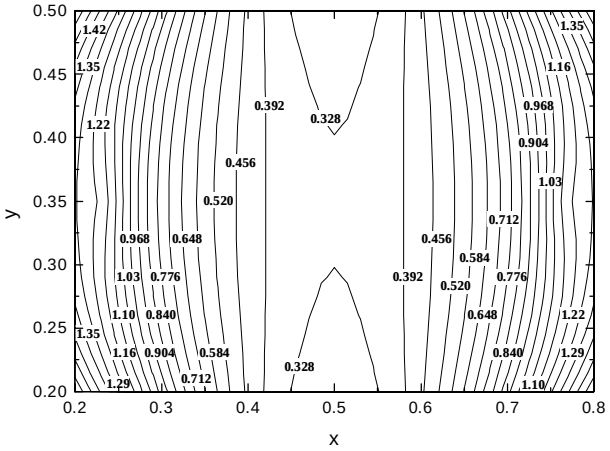

Fig. 5 - Stadium-shaped pipe. Coherent Laslett coefficient (1st normal mode).

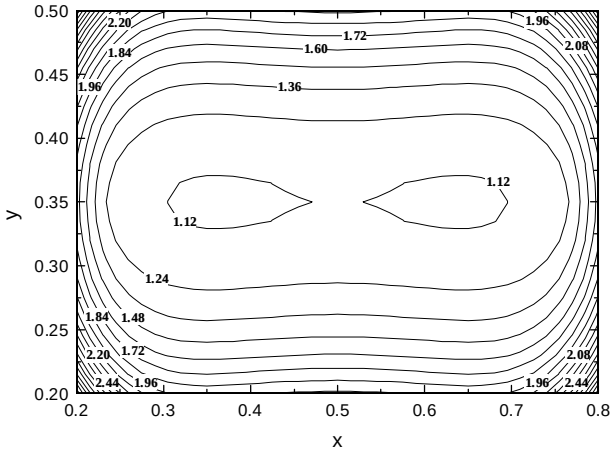

Fig. 6 - Stadium-shaped pipe. Coherent Laslett coefficient (2nd normal mode).