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BEAM INDUCED LOSSES IN THE ELECTRON LINAC

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It will probably not surprise the reader that early discussions on the design of linear accelerators did not consider the effects of beam loading upon achievable energy gain, doubtless for the reason that the then state of technology would not permit more than a slight beam current. Subsequently, with the improvement of injection systems (about 1950), departures from no-load energy as a consequence of beam current aroused interest in the precise analysis of beam loading.

At first it was supposed the differential power loss in the waveguide was owing to joulean attenuation (2I) and beam absorption of power (equal to the rate of energy gain of the accelerated particles), which can be expressed in a general way, for a synchronous wave by the power diffusion equation

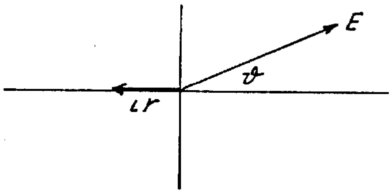
$$\frac{dP}{dz} = -2IP - IE\cos\vartheta \tag{1}$$

where by ϑ was meant the phase relation between the RF field and the beam, the amplitude of the electric field being given by $E(z)^2 = 2IP(z)r$. The resulting solution, carried on to the energy gain equation, with boundary conditions $P = P_0$, $z = 0$ is

$$V = \sqrt{2IP_0r}\,\frac{1-e^{-IL}}{I} - irL\cos\vartheta\left(1-\frac{1-e^{-IL}}{IL}\right) \tag{2}$$

which is clearly incorrect, and wrong for the reason that the energy gain in the case of nearly vanishing beam current appears not to depend on the injection phase, which it clearly does. (1)

Examination of the vector diagram,



where E is the applied RF field and ir the induced field, shows that the appropriate differential equations are:

$$\frac{d(E\cos\vartheta)}{dz} = -I(E\cos\vartheta + ir)$$
$$\frac{d(E\sin\vartheta)}{dz} = -I(E\sin\vartheta) \tag{3}$$

whose solution subject to the boundary conditions at $z = 0$

$$E\cos\vartheta + ir = E_0\cos\vartheta_0 + ir$$
$$E\sin\vartheta = E_0\sin\vartheta_0 \tag{4}$$

are:

$$E\cos\vartheta = E_0\cos\vartheta_0 e^{-Iz} - ir(1-e^{-Iz})$$
$$E\sin\vartheta = E_0\sin\vartheta_0 e^{-Iz} \tag{5}$$

or,

$$E^2 = (E_0e^{-Iz})^2 - 2irE_0\cos\vartheta_0 e^{-Iz}(1-e^{-Iz}) + (ir)^2(1-e^{-Iz})^2$$
$$\tan\vartheta = \frac{\tan\vartheta_0}{1-\frac{ir}{E_0\cos\vartheta_0}(e^{Iz}-1)} \tag{6}$$

from which the energy gain is, therefore,

$$V = \int_0^L E_0\cos\vartheta_0 e^{-Iz} - ir(1-e^{-Iz})\cdot dz$$
$$= E_0\cos\vartheta_0\,\frac{1-e^{-IL}}{I} - irL\left(1-\frac{1-e^{-IL}}{IL}\right) \tag{7}$$

The power flux, from eq (6),

$$P(z) = P_0e^{-2Iz} + \sqrt{\frac{2P_0r}{I}}\,i\cos\vartheta_0(e^{-Iz}-1)e^{-Iz} + \frac{i^2r}{2I}(e^{-Iz}-1)^2 \tag{8}$$

which is clearly the input power attenuated, as diminished by beam absorption, and increased by re-radiation into the structure. (2)

These trifling considerations can be extended to the case of the constant (electric) gradient waveguide without difficulty. The in-phase component of the RF wave is beam-loaded in this case, but by parameter design neither component of field is attenuated by joulean losses,

$$\frac{d(E\cos\vartheta)}{dz} = -Iir = -\frac{I_0ri}{(1-2I_0z)}$$
$$\frac{d(E\sin\vartheta)}{dz} = 0 \tag{9}$$

The solution of these equations subject to the boundary conditions at $z = 0$; $E = E_0$, $\vartheta = \vartheta_0$ are:

$$E\cos\vartheta = E_0\cos\vartheta_0 + \frac{ir}{2}\ln(1-2I_0z)$$
$$E\sin\vartheta = E_0\sin\vartheta_0 \tag{10}$$

or,

$$E^2 = E_0^2 + E_0ir\cos\vartheta_0\ln(1-2I_0z) + \left[\frac{ir}{2}\ln(1-2I_0z)\right]^2$$
$$\tan\vartheta = \frac{\tan\vartheta_0}{1+ir\ln(1-2I_0z)/2E_0\cos\vartheta_0} \tag{11}$$

corresponding to a beam loaded power flux,

$$P(z) = P_0(1-2I_0z) + P_0(1-2I_0z)ir\cos\vartheta_0\ln(1-2I_0z) + \frac{1-2I_0z}{2I_0r}\left[\frac{ir}{2}\ln(1-2I_0z)\right]^2 \tag{12}$$

of which the physical interpretation is the same as previously given. The energy gain is therefore, from eq (10),

$$V = \int_0^L E_0\cos\vartheta_0 + \frac{ir}{2}\ln(1-2I_0z)\cdot dz$$
$$= E_0L\cos\vartheta_0 - \frac{ir}{4I_0}\left[2I_0L + (1-2I_0L)\ln(1-2I_0L)\right] \tag{13}$$

These well-known results are quoted to emphasize that the power diffusion equation in the form

$$\frac{dEe^{j\vartheta}}{dz} = -I(Ee^{j\vartheta} + ir) \tag{14}$$

with boundary conditions $Ee^{j\vartheta} + ir = E_0e^{j\vartheta_0} + ir$ at $z = 0$ has the solution

$$Ee^{j\vartheta} = E_0e^{j\vartheta_0 - Iz} + ir(e^{-Iz}-1) \tag{15}$$

That is, the beam loading equation includes re-radiation in the original differential equation and is not an effect to be added as a later correction. As the reader may find it of interest to review development of the theory of beam loading in the technical literature, ref (3) lists the principal discussions.

When the bunched beam traverses the RF structure it induces an RF signal in the structure dependent on the shunt impedance presented to it by the structure. But in a periodic waveguide the situation is more complicated than in a singly resonant cavity. The energy which is radiated into the structure travels along the waveguide, so that the E-field launched by the beam, which is π-radians out-of-phase with the beam by Le Chatelier's principle, is attenuated by the structure,

$$\frac{dE}{dz} = -I(E + ir) \tag{16}$$

the solution of which is, with the boundary conditions $E = 0$, $z = 0$:

$$E(z) = -ir(1-e^{-Iz}) \tag{17}$$

The same results can be derived on a power basis,

$$dP/dz = -2IP - iE \tag{18}$$

the solution of which is, with boundary conditions $P = 0$, $z = 0$:

$$P(z) = \frac{i^2r}{2I}(1-e^{-Iz})^2 \tag{19}$$

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In the above it has been assumed that the beam travels synchronously (in a constant impedance structure) with the induced wave and that the beam is highly relativistic. If the interaction of the beam and the induced wave varied the motion of the bunch the problem would become very complicated (as it does, for example, in a klystron 'catcher' cavity). It has also been assumed that the system is at steady state. If the wave and beam are not synchronous the induced power is lessened by interference in the proportion

$$\frac{P}{P_o} = \left[\frac{\eta \sin(2\pi L/\lambda)}{\eta(2\pi L/\lambda)}\right]^2 \tag{20}$$

where P_o is the synchronous power and η is the fractional deviation from synchronism. Thus, if v_p is the phase velocity of the wave and v_e that of the electron, $v_p/v_e = 1 + \eta$, or if there is a frequency error $\eta = (v_p/v_g)\,(\delta\omega/\omega)$, where v_g is the group (energy) velocity at the specified phase velocity.

In the transient case the induced power is evidentally

$$P(z,t) = \frac{i^2 r}{2I}\left(e^{-I v_g t} - 1\right)^2 \qquad z > v_g t \tag{21}$$

When $v_g t = z$ the previous (upstream) part of the guide is in steady state and eq (19) describes the beam induced power at that location.

Elementary as the foregoing comments appear they have not been accepted without expository criticism. The earliest of these was a consideration of the manner in which a signal propagates in a lossey, dispersive waveguide (4)

Given an ideal pulse, for example

$$\begin{aligned} f(t) &= \cos\omega_o t & (-T<t<T) \\ &= 0 & (T<t<-T) \end{aligned} \tag{22}$$

and its Fourier spectrum

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\,e^{-j\omega t}\,dt = \frac{\sin(\omega_o-\omega)T}{\omega_o-\omega} + \frac{\sin(\omega_o+\omega)T}{\omega_o+\omega}$$

we can, in principle, obtain the form of the pulse at some distance into the waveguide by forming the inverse transform with a modified kernel (the system transfer function),

$$f(z,t) = \frac{1}{2\pi}\int_0^{\infty} F(\omega)\,e^{j\omega t - \gamma z}\,d\omega \tag{23}$$

Physically this is merely the sum of the frequency components which have arrived at a location z and time t, each component travelling with its own phase velocity and attenuation.

The real and imaginary components of the propagation constant ($\gamma = I + j\beta$ in the customary notation) are not usually independent and are connected by the Kramers-Kronig relation which may be written in the form (5)

$$\beta_c = \frac{1}{\pi}\int_0^{\infty}\frac{dI}{d\omega}\ln\left|\frac{\omega+\omega_c}{\omega-\omega_c}\right|d\omega \tag{24}$$

where β_c is the phase shift at frequency $\omega_c/2\pi$. This integral cannot, in general, be evaluated readily, except in the simplest cases and only slight results have been obtained by numerical methods. Also, it is not apparent that Bode's numerical method is valid for periodic structures since a transmission system with infinite loss (no transmission) at a real frequency is not analytic in the right half p-plane.

The customary method of attacking such a problem is the assumption of equalized amplitude and linear phase in the essential range of frequencies contained in the transmitted pulse; although these are necessary and sufficient conditions for undistorted transmission they are not physically realizable. Then a mid-band signal (ω_o) can be shown to be given by (6)

$$f(z,t) = \frac{K}{\pi}\left[Si\,\frac{\Delta\omega}{2}\left(t-\frac{z}{v_g}\right) - Si\,\frac{\Delta\omega}{2}\left(t+T-\frac{z}{v_g}\right)\right]\cos\omega_o t \tag{25}$$

where K is the amplitude coefficient, $\Delta\omega$ is the system band-width and Si(x) is the sine integral function. (7)

With the above sort of solution for the pulse propagation the energy gain of a particle sychronized with a point on the wave will be

$$V = \int f(z,t)\,dz \tag{26}$$

from which it will be seen the effect is primarily to broaden the energy spectrum.

The conventional analysis of beam interaction presented earlier is based on the assumption that the only mode in the structure is the intended accelerating mode. But the RF structure of the beam will of course react with all the higher order (including non-propagating or evanescent) modes of the structure. In the past this has been neglected, both because it was trivial in magnitude at the beam currents then used and because of the apparently formididable calculations involved. Nevertheless, several investigators have attacked the problem and have presented quantitative estimates of energy losses involved and some experimental data. (8)

One formulation of this calculation is based upon a generalized description of all the oscillatory modes of a cavity and the implied coupling between the beam bunches and the fields of each mode(9). The catalog of modes must extend up to indices until such perturbations as the beam aperture is involved (not into the optical region, for example, because the mode description will then no longer matter). Another method of performing this calculation is an 'optical' one, which transcends the details of the modes of the structure. (10) A rather thorough treatment of this problem has been given by E. Keil. (11)

Several laboratories have reported beam data with sufficient accuracy to be useful for substantiating the above conclusions. NBS has presented data on the energy loss of a beam transiting an otherwise unexcited waveguide with short beam pulses (10 nsec) of various currents (total charge). (12) This experiment is adequately justified, to the precision which the factors involved are known, by the analysis of the transient operation of an electron linear accelerator. (13)

The above data for almost trivial charges, reported in the above case, have been supplemented by single RF burst data from ISIR, Osaka University. (14) In this latter case the energy gain of the beam pulse (for fixed RF input power) is given for a 38 psec pulse (FWHM) for various charges/pulse up to 16 nC. These data, too, are closely justified by the transient analysis of ref. (13).

For the purpose of numerically examining the data presented in the above two papers it is necessary to know the properties of the accelerating waveguide, which was not presented in either paper. These are listed below: (15)

	NBS*	ISIR, Osaka**
Operating freq.f, ($v_p = c$) mcs.	1300	1300
Shunt impedance, r, $M\Omega$/m	35	40
Figure of merit, Q($2\pi/3$-mode)	19,200	19,000
Attenuation length, $2I_oL$, nep	0.745	0.834
Length, L, meters	2.5	3.0
(Initial) atten. coeff. I_o nep/m	0.149	0.0944
(Initial) norm. group vel.v_g/c	0.0042	0.0075
Fill time, τ usec	2.0	1.96
No load energy, V_o, MeV	21.3(10MW)	31(18MW)

* Constant impedance design
** Constant (electric) gradient design

For the purpose of demonstrating the transient operation energy gain of an accelerator example will be taken of the Osaka machine. In Fig. 1, the theoretical energy gain during a pulse for a specified beam current is shown, based upon the equations of ref (13). Clearly, these data may be expressed in a simpler diagram, Fig. 2 (which also shows some unpublished experimental data from that linac). One can

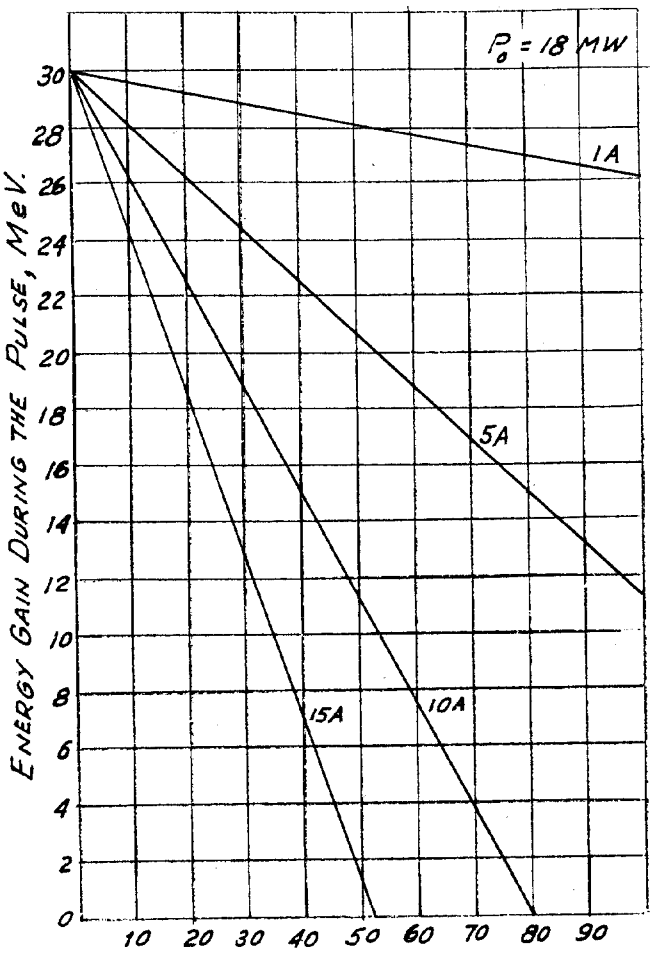

FIG.1 PULSE DURATION, nsec.

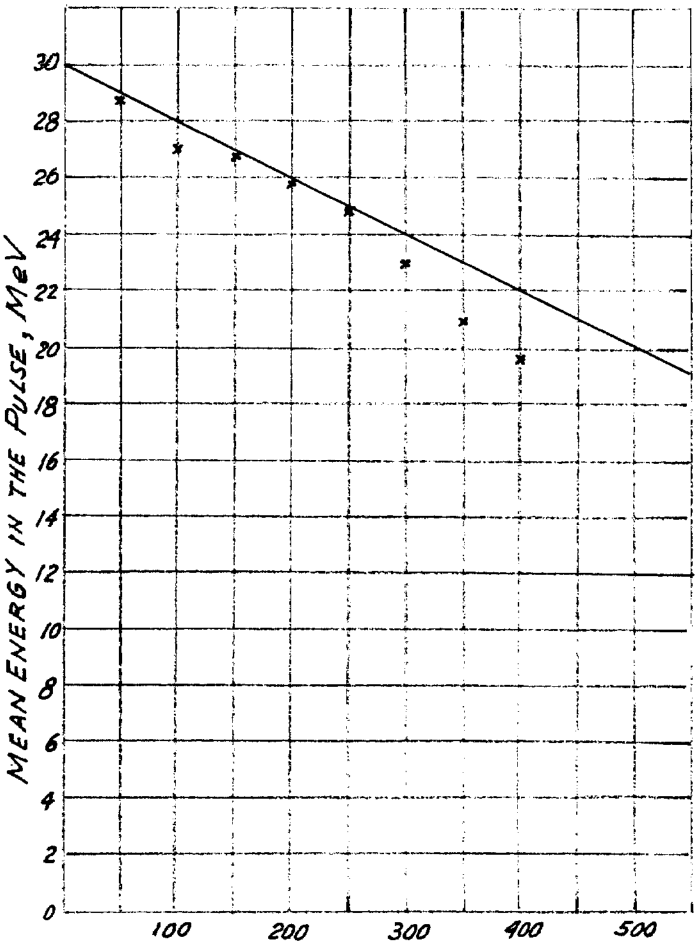

FIG.2 PULSE CHARGE, nC.

conclude that the beam-loaded calculations are sufficiently accurate for engineering purposes, and while the consequences of re-radiation are of little effect in existing accelerators, the beam currents intended by FEL designers and high current superconducting devices will require consideration of re-radiation effects.

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