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FREE ELECTRON LASER INSTABILITY FOR A RELATIVISTIC ELECTRON BEAM IN A HELICAL WIGGLER FIELD

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Summary

The free electron laser instability is investigated for a relativistic annular or solid electron beam propagating through a helical wiggler magnetic field. It is assumed that $\nu/\gamma_b \ll 1$, where ν is Budker's parameter. In this regard, the stability analysis utilizes the vacuum transverse electric and transverse magnetic waveguide modes as a convenient basis to represent the electromagnetic field perturbations. Stability properties are investigated including the important influence of (a) finite beam geometry in the radial direction, (b) positioning of the beam radius relative to the outer conducting wall radius (R_c) and (c) finite wiggler amplitude. All of these effects are shown to have an important influence on detailed stability behavior.

Introduction

In recent years, the free electron laser instability[1-5] has been extensively investigated with particular emphasis on the implications for intense microwave generation. In this paper, we develop a simple description of the free electron laser instability in the tenuous beam limit, including the important influence of finite radial geometry.

The equilibrium configuration consists of a relativistic (annular or solid) electron beam propagating in the combined helical wiggler and uniform axial guide fields described by

$$\underset{\sim}{B}^o = -\delta B \cos(\theta - k_o z)\, \hat{\underset{\sim}{e}}_r + \delta B \sin(\theta - k_o z)\hat{\underset{\sim}{e}}_\theta + B_o \hat{\underset{\sim}{e}}_z, \tag{1}$$

where B_o and δB are constants, and k_o is the axial wave number of the helical wiggler field. In Eq. (1), cylindrical coordinates (r, θ, z) are used. It is assumed that

$$\nu/\gamma_b \ll 1 \tag{2}$$

where $\nu = Ne^2/mc^2$ is Budker's parameter, N is the number of beam electrons per unit axial length, c is the speed of light in vacuo, and $-e$ and m are the electron charge and rest mass, respectively. Consistent with the low-density assumption in Eq. (2), we neglect the influence of the weak equilibrium self-electric and self-magnetic fields that are produced by the lack of equilibrium charge and current neutrality. Moreover, the perturbed electromagnetic fields are approximated by the vacuum waveguide fields.

Linearized Vlasov-Maxwell Equations

The stability analysis is carried out within the framework of the linearized Vlasov-Maxwell equations. We adopt a normal-mode approach in which all perturbations are assumed to vary with time and space according to

$$\delta\psi(\underset{\sim}{x},t) = \sum_{\ell=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{\psi}_\ell^{(n)}(r) \exp\left\{ i\left[\ell\theta + (k + nk_o)z - \omega t\right]\right\}, \tag{3}$$

where $\text{Im}\,\omega > 0$. Here ω is the complex eigen frequency $k + nk_o$ is the axial wavenumber, and ℓ and n are integers. It is also assumed that the wave perturbations are close to resonance with

$$|\omega - (k + nk_o) V_b| \ll \omega_o, \omega_c, \tag{4}$$

where $\omega_o = k_o V_b$, $\omega_c = eB_o/\gamma_b mc$ and $V_b = (\gamma_b^2 - 1)^{1/2} c/\gamma_b$.

The Maxwell equations for the perturbed electric and magnetic fields amplitude can be expressed as

$$\underset{\sim}{\nabla} \times \hat{\underset{\sim}{E}}(\underset{\sim}{x}) = i(\omega/c)\hat{\underset{\sim}{B}}(\underset{\sim}{x}),$$
$$\underset{\sim}{\nabla} \times \hat{\underset{\sim}{B}}(\underset{\sim}{x}) = (4\pi/c)\hat{\underset{\sim}{J}}(\underset{\sim}{x}) - i(\omega/c)\hat{\underset{\sim}{E}}(\underset{\sim}{x}), \tag{5}$$

where

$$\hat{\underset{\sim}{J}}(\underset{\sim}{x}) = -e\int d^3p\, \underset{\sim}{v}\, \hat{f}_b(\underset{\sim}{x},\underset{\sim}{p}) \tag{6}$$

is the perturbed current density. In Eq. (6),

$$\hat{f}_b(\underset{\sim}{x},\underset{\sim}{p}) = e\int_{-\infty}^{0} d\tau \exp(-i\omega\tau)\left[\hat{\underset{\sim}{E}} + \frac{\underset{\sim}{v}' \times \hat{\underset{\sim}{B}}}{c}\right]\cdot \frac{\partial}{\partial \underset{\sim}{p}'} f_b^o \tag{7}$$

is the perturbed distribution function, and $\tau = t' - t$, and $d\underset{\sim}{x}'/dt' = \underset{\sim}{v}'$ and $d\underset{\sim}{p}'/dt' = -e\underset{\sim}{v} \times \underset{\sim}{B}^o/c$, with initial conditions $\underset{\sim}{x}'(t' = t) \cong \underset{\sim}{x}$ and $\underset{\sim}{v}'(t' = t) = \underset{\sim}{v}$. Within the context of Eqs. (2) and (4), Eq. (7) can be approximated by

$$\hat{f}_b(\underset{\sim}{x},\underset{\sim}{p}) = -\frac{ie}{\omega}\int_{-\infty}^{0} d\tau \exp(-i\omega\tau)\left\{2\left[\gamma m i\omega(\underset{\sim}{v}'\hat{\underset{\sim}{E}}) - p_z(\underset{\sim}{v}' \cdot \frac{\partial}{\partial z}\hat{\underset{\sim}{E}})\right]\frac{\partial}{\partial p_\perp^2} f_b^o + (\underset{\sim}{v}' \cdot \frac{\partial}{\partial z}\hat{\underset{\sim}{E}})\frac{\partial}{\partial p_z} f_b^o\right\}, \tag{8}$$

where γmc^2 is the energy and f_b^o is the equilibrium electron distribution function.

To lowest order, for $\delta B/B_o \ll 1$ and $k_o v_z$ well removed from cyclotron resonance ($\omega_o \neq \omega_c$), the axial motion of the electron orbit is free-streaming

$$z' = z + (p_z/\gamma m)(t' - t). \tag{9}$$

Moreover, within the context of Eq. (4), on the right-hand side of Eq. (8) we approximate the radial and azimuthal motion of the electron orbit by[4]

$$v_r' = v_z \frac{\omega_c}{\omega_o - \omega_c} \frac{\delta B}{B_o} \cos(\theta - k_o z - k_o v_z \tau), \quad (10)$$

and

$$v_\theta' = -v_z \frac{\omega_c}{\omega_o - \omega_c} \frac{\delta B}{B_o} \sin(\theta - k_o z - k_o v_z \tau). \quad (11)$$

Finally, since the oscillatory modulation of the radial and azimuthal orbit is assumed to be small amplitude, we approximate $r' \simeq r$ and $\theta' \simeq \theta$ in the arguments of the perturbations amplitudes on the right-hand side of Eq. (8).

Substituting Eqs. (9) - (11) into Eq. (8), we obtain

$$\hat{f}_b(\underset{\sim}{x}, \underset{\sim}{p}) = \frac{ie}{\omega} \sum_{\ell,n} \frac{\exp\left\{ i\left[\ell\theta + (k + nk_o)z\right]\right\}}{\omega - (k + nk_o)v_z} \left\{ \lambda_n \beta_z \hat{E}^{(n)}_{z,\ell}(r) + \Lambda\, \lambda_{n-1} \left[\hat{E}^{(n-1)}_{r,\ell+1}(r) - i\, \hat{E}^{(n-1)}_{\theta,\ell+1}(r)\right]\right\} \quad (12)$$

where

$$\lambda_{n'}(\underset{\sim}{p}, \omega, k) = 2\left[\gamma m\omega - (k + n'k_o)p_z\right] \cdot \frac{\partial f_b^o}{\partial p_\perp^2} + (k + n'k_o) \frac{\partial f_b^o}{\partial p_z} \quad (13)$$

and

$$\Lambda = \frac{e\delta B}{2\gamma_b m c^2 k_o} \frac{\omega_o}{\omega_o - \omega_c}. \quad (14)$$

In Eq. (12), the term proportional to λ_n is the longitudinal portion of the perturbed disbribution function. Similarly, the terms proportional to λ_{n-1} in Eq. (12) are the transverse electromagnetic portion.

From Poisson equation

$$\underset{\sim}{\nabla} \cdot \hat{\underset{\sim}{E}}(\underset{\sim}{x}) = 4\pi\hat{\rho}(\underset{\sim}{x})$$

and the Maxwell Eq. (5), we obtain the differential equation

$$\left[\nabla_\perp^2 + \frac{\omega^2}{c^2} - (k + nk_o)^2\right] \hat{E}^{(n)}_{z,\ell}(r) = \frac{4\pi i\,(k + nk_o)}{\gamma_b^2} \hat{\rho}_\ell^{(n)}(r) \quad (15)$$

for the axial (longitudinal) component of the perturbed electric field. In Eq. (15), $\hat{\rho}_\ell^{(n)}(r)$ is the perturbed charge density, $\nabla_\perp^2$ denotes $(1/r)(\partial/\partial r)(r\,\partial/\partial r) - \ell^2/r^2$, and use has been made of $\hat{J}^{(n)}_{z,\ell} \simeq V_b \hat{\rho}_\ell^{(n)}$. In the tenuous beam limit consistent with Eq. (2), the transverse component of the perturbed field in Eq. (12) can be approximated by the vacuum waveguide fields. In this context, the present stability analysis utilizes the vacuum transverse electric (TE) and transverse magnetic (TM) waveguide mode as a convenient basis to represent the general electromagnetic field perturbation determined from

$$\left[\frac{\omega^2}{c^2} - (k + nk_o - k_o)^2\right] \hat{\underset{\sim}{E}}^{(n-1)}_{\perp,\ell+1}(\underset{\sim}{x}) = \underset{\sim}{\nabla}_\perp \frac{\partial}{\partial z} \hat{E}^{(n-1)}_{z,\ell+1}(\underset{\sim}{x}) - i\, \frac{\omega}{c} \hat{\underset{\sim}{e}}_z \times \underset{\sim}{\nabla}_\perp \hat{B}^{(n-1)}_{z,\ell+1}(\underset{\sim}{x}). \quad (16)$$

Neglecting the perturbed current density, the vacuum waveguide modes can be expressed as

$$\hat{B}^{(n-1)}_{z,\ell+1}(r) = b_{\ell+1,s}\; J_{\ell+1}(\alpha_{\ell+1,s} r/R_c),$$

$$\hat{E}^{(n-1)}_{r,\ell+1}(r) - i\, \hat{E}^{(n-1)}_{\theta,\ell+1}(r) = -\frac{\omega R_c}{c\alpha_{\ell+1,s}} b_{\ell+1,s} \cdot J_\ell(\alpha_{\ell+1,s}\, r/R_c), \quad (17)$$

for the TE mode and

$$\hat{E}^{(n-1)}_{z,\ell+1}(r) = \epsilon_{\ell+1,s}\, J_{\ell+1}(\beta_{\ell+1,s}\, r/R_c),$$

$$\hat{E}^{(n-1)}_{r,\ell+1}(r) - i\, \hat{E}^{(n-1)}_{\theta,\ell+1}(r) = i\, \frac{(k + n\,k_o - k_o)R_c}{\beta_{\ell+1,s}} \cdot \epsilon_{\ell+1,s}\, J_\ell(\beta_{\ell+1,s} r/R_c) \quad (18)$$

for the TM mode. In Eqs. (17) and (18), R_c is the radius of the grounded conducting wall, $b_{\ell+1,s}$ and $\epsilon_{\ell+1,s}$ are constants, $J_{\ell'}(x)$ is the Bessel function of the first kind of order ℓ', and $\alpha_{\ell+1,s}$ and $\beta_{\ell+1,s}$ are the sth roots of $J'_{\ell+1}(\alpha_{\ell+1,s}) = 0$ and $J_{\ell+1}(\beta_{\ell+1,s}) = 0$, respectively. Here the prime (') denotes $J_{\ell+1}(x) = (d/dx)J_{\ell+1}(x)$. After some straight forward algebraic manipulation of Eqs. (5), (17) and (18), we obtain

$$\left[\frac{\omega^2}{c^2} - (k + nk_o - k_o)^2 - \frac{\alpha^2_{\ell+1,s}}{R_c^2}\right] b_{\ell+1,s}\, J_{\ell+1}\left(\frac{\alpha_{\ell+1,s} r}{R_c}\right) = -\frac{4\pi}{rc}\left\{\frac{\partial}{\partial r}\left[r\, \hat{J}^{(n-1)}_{\theta,\ell+1}(r)\right] - i\,(\ell+1)\, \hat{J}^{(n-1)}_{r,\ell+1}(r)\right\}, \quad (19)$$

for the TE mode and

$$\left[\frac{\omega^2}{c^2} - (k + nk_o - k_o)^2 - \frac{\beta^2_{\ell+1,s}}{R_c^2}\right] \epsilon_{\ell+1,s}\, J_{\ell+1}\left(\frac{\beta_{\ell+1,s}\, r}{R_c}\right) = \frac{4\pi}{rc}\left\{\frac{\partial}{\partial r}\left[r\, \hat{J}^{(n-1)}_{r,\ell+1}(r)\right] + i\,(\ell+1)\, \hat{J}^{(n-1)}_{\theta,\ell+1}(r)\right\}, \quad (20)$$

for the TM mode. Equations (15), (19) and (20), combined with Eqs. (12) and (13), constitute one of the main results of this paper and can be used to investigate free electron laser stability properties for a broad range of physical parameters.

Relativistic Annular Electron Beam

As first application, we consider the free electron laser instability for a relativistic annular electron beam described by[6]

$$f_b^o = \frac{n_o R_o \gamma_b m v_o^2}{2\Delta} U\left[(C_\perp - \gamma_b^2 m^2 v_o^2)^2 - \Delta^2\right]$$

$$\delta(C_h - C_o)\delta(C_z - \gamma_b m V_b), \qquad (21)$$

where $C_\perp$, C_h and C_z are the transverse, helical and axial invarients in the combined transverse wiggler and uniform axial guide field [Eq. (1)], and $U(x)$ is the Heaviside step function. Substituting Eq. (21) into Eqs. (12) and (13), and assuming that thickness of the annular beam is much smaller than the beam radius R_o, we obtain

$$\hat{J}_{\theta,\ell+1}^{(n-1)} = i\,\hat{J}_{r,\ell+1}^{(n-1)} = i\,c\hat{\rho}_\ell^{(n)} = \frac{\delta(r-R_o)}{R_o}\frac{\Lambda c^2}{4\pi\omega}\sigma_\ell, \qquad (22)$$

$$\sigma_\ell = \chi_{n,n}\beta_z \hat{E}_{z,\ell}^{(n)} + \Lambda\chi_{n,n-1}\left[\hat{E}_{r,\ell+1}^{(n-1)} - i\,\hat{E}_{\theta,\ell+1}^{(n-1)}\right], \qquad (23)$$

and

$$\chi_{n,n'} = \frac{2\nu}{\gamma_b}\frac{\omega^2 - (k+nk_o)(k+n'k_o)c^2}{\left[\omega - (k+nk_o)V_b\right]^2}. \qquad (24)$$

Making use of Eqs. (17), (22) and (23), and defining

$$q_n^2 = (k+nk_o)^2 - \omega^2/c^2,$$

the eigen value equation (15) for the longitudinal perturbation can be expressed as

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{\ell^2}{r^2} - q_n^2\right)\hat{E}_{z,\ell}^{(n)}(r)$$

$$= \frac{\delta(r-R_o)}{\gamma_b^2 R_o}\left[\chi_{n,n}\hat{E}_{z,\ell}^{(n)} - \Lambda\frac{(k+nk_o)R_c}{\alpha_{\ell+1,s}}\right.$$

$$\left.\cdot\, b_{\ell+1,s}\chi_{n,n-1}J_\ell\left(\frac{\alpha_{\ell+1,s}r}{R_c}\right)\right] \qquad (25)$$

for the TE mode. Since the right-hand side of Eq. (25) vanishes except at $r = R_o$, the solution to Eq. (25) is given by

$$\hat{E}_{z,\ell}^{(n)}(r) = \begin{cases} A\,I_\ell(q_n r), & 0 \le r < R_o, \\ B\left[I_\ell(q_n r) - K_\ell(q_n r)I_\ell(q_n R_c)/K_\ell(q_n R_c)\right], & R_o < r < R_c, \end{cases} \qquad (26)$$

where $I_\ell(x)$ and $K_\ell(x)$ are the modified Bessel functions of the first and second kind, respectively of order ℓ. From Eq. (25) and (26), we obtain

$$h(R_o)\hat{E}_{z,\ell}^{(n)}(R_o) = \frac{I_\ell(q_nR_c)/I_\ell(q_nR_o)}{I_\ell(q_nR_c)K_\ell(q_nR_o) - I_\ell(q_nR_o)K_\ell(q_nR_c)}$$

$$\cdot\,\hat{E}_{z,\ell}^{(n)}(R_o) = \frac{1}{\gamma_b^2}\left[-\chi_{n,n}\hat{E}_{z,\ell}^{(n)}(R_o) + \Lambda\frac{(k+nk_o)R_c}{\alpha_{\ell+1,s}}\right.$$

$$\left.\cdot\, b_{\ell+1,s}\chi_{n,n-1}J_\ell\left(\frac{\alpha_{\ell+1,s}R_o}{R_c}\right)\right] \qquad (27)$$

from Eqs. (25) and (26).

Substituting Eqs. (22) and (23) into Eq. (19), multiplying Eq. (19) by $r\,J_{\ell+1}(\alpha_{\ell+1,s}\,r/R_c)$, and integrating from $r = 0$ to $r = R_c$, we obtain two homogeneous equations relating the ampltudes $E_{z,\ell}^{(n)}(R_o)$ and $b_{\ell+1,s}$. The condition for a nontrival solution is that the determinant of the coefficients of $E_{z,\ell}^{(n)}(R_o)$ and $b_{\ell+1,s}$ be equal to zero. Setting the determinant equal to zero, we find, after some algerbraic manipulation, that the TE mode dispersion relation is given by

$$\left\{\left[\frac{\omega^2}{c^2} - (k+nk_o-k_o)^2 - \frac{\alpha_{\ell+1,s}^2}{R_c^2}\right]\left[\omega - (k+nk_o)V_b\right]^2 - \frac{\nu}{\gamma_b^3 h}\left[(k+nk_o)^2c^2 - \omega^2\right]\right\} = 4\Lambda^2\frac{\nu}{\gamma_b R_c^2}\left[(k+nk_o)\right.$$

$$\left.\cdot\,(k+nk_o-k_o)c^2 - \omega^2\right]Q_{\ell s}^E \qquad (28)$$

where the coupling coefficient

$$Q_{\ell s}^E(R_o/R_c) = \frac{\alpha_{\ell+1,s}^2}{\alpha_{\ell+1,s}^2 - (\ell+1)^2}\left[\frac{J_\ell(\alpha_{\ell+1,s}R_o/R_c)}{J_{\ell+1}(\alpha_{\ell+1,s})}\right]^2. \qquad (29)$$

Similarly, we can obtain the dispersion relation[4] for the TM mode. The dispersion relation in Eq. (28) can be used to investigate stability properties for a broad range of system parameters.

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