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ENERGY GAIN IN RESONANTLY EXCITED, SLOW-WAVE STRUCTURES

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INTRODUCTION Discussions of relatively complicated, resonantly excited, periodic structures have appeared in the recent accelerator literature for proposed beam-line (acceleration) waveguides. This paper presents results of an analysis of effects on, and energy gain in, beam-loaded cavities.

While it is the ultimate intention of this paper to investigate a quite general class of resonant structures with beam-loaded applications, it seems desirable to consider first the case of a simple, singly resonant cavity, as this case usually is not treated in the technical literature of electronic devices.

An adequate description of the behavior of a resonant cavity is given in terms of three constants of the cavity; the resonant frequency (ω_0), the quality factor (Q_o) and the shunt impedance (R_o). All three constants are determined from the solution of the wave equation in the geometry of the cavity (ignoring the coupling mechanism). It is scarcely necessary to comment on the significance of these constants but that microwave engineers customarily consider Q as the ratio of the stored energy in the circuit to the energy loss in one radian rather than the sharpness of frequency response (to which it is demonstrably equivalent), ie.,

$$Q = \frac{\omega W}{P} \tag{1}$$

and R as the ratio of the electric line integral (squared) along a specified path (the beam interaction trajectory) to the power input to the cavity,

$$R = \frac{(\int E \cdot dz)^2}{2P} \tag{2}$$

The factor 2 arises because it is a convention with accelerator engineers to express $V = \int E .dz$ as a peak voltage in the cavity.

The properties of the cavity stated above are somewhat impractical as they are derived from cavity fields without reference to how the RF energy got into the cavity. When the coupling does not repartition the field distribution in the cavity,

$$\frac{R_L}{Q_L} = \frac{R_o}{Q_o} \tag{3}$$

where the subscripts indicate 'loaded' and 'unloaded' by coupling, respectively. (Rearrangement of the field pattern in the cavity is generally signalized by resonant frequency perturbation.)

It is well-known from circuit theory that without beam loading (1):

$$\frac{1}{Q_L} = \frac{1}{Q_o} + \frac{1}{Q_E} \tag{4}$$

and the ratios

$$\frac{Q_o}{Q_E} = \beta \qquad \frac{Q_o}{Q_L} = 1 + \beta \tag{5}$$

may be defined where Q_E characterizes additional losses owing to drive line coupling and β is the coupling coefficient (VSWR, σ, at resonance or its reciprocal). Thus, eq (3) may be written

$$\frac{R_o}{R_L} = \frac{Q_o}{Q_L} = 1 + \beta \tag{6}$$

That is, if the coupling coefficient is adjusted to β the loaded shunt resistance will be $R_L = R_o/(1+\beta)$ and similarly for the loaded Q. If a load resistance R be placed across the output terminals of the network (ie. along the beam interaction gap) the input coupling coefficient (VSWR) will be, where Z_o is the characteristic impedance of the drive line and β the no-load coupling coefficient,

$$\beta' = \frac{1}{Z_o}\left(\frac{R_L R}{R_L + R}\right) = \beta \frac{R}{R_L + R} \tag{7}$$

The Q of the cavity when loaded by the beam (Q_o') is given by

$$\frac{1}{Q_o'} = \frac{1}{Q_o} + \frac{1}{Q_B} = \frac{P_r}{\omega W} + \frac{P_B}{\omega W} = \frac{P_o}{\omega W} \tag{8}$$

where $Q_o = \omega W/P_r$ is the unloaded Q of the cavity, $Q_B = \omega W/P_B$ is the equivalent Q of the beam, P_r is the power delivered to the structure, P_B is the power delivered to the beam and W is the stored energy in the cavity, and $P_o = P_r + P_B$.

Additionally, when a cavity of no load shunt impedance R_o and no load Q_o is loaded with a resistance R, the apparent Q_o' will be lowered, such that (2)

$$\frac{R}{Q_o'} = \frac{R_o + R}{Q_o} \tag{9}$$

or

$$Q_o' = \frac{R}{R_o + R} Q_o \tag{10}$$

In the steady state the power input to the cavity (P_o) is determined by the coupling coefficient (β)

$$P_o = P_i(1 - |\rho|^2) = \frac{4\beta}{(1+\beta)^2} P_i \tag{11}$$

where $\rho = (\beta - 1)/(\beta + 1)$ is the voltage reflection coefficient and P_i is the incident (available) power from the transmission line. This power is, in the steady state, partitioned into power delivered to the RF structure as joulean losses (P_r) and power delivered to the beam (P_B),

$$P_o = P_r + P_B \tag{12}$$

The electric field in the cavity produced by oscillations of stored energy is described by eq (2). This electric line integral (voltage) is the peak voltage across the gap; it is not the energy gain of a particle transiting the gap for the reason that this voltage only exists instantaneously and the particles of the beam require a finite time to cross the gap. Integration across the gap is ordinarily a matter of considerable complexity, but when the particle velocity is fixed (highly relativistic) the calculation is simple,

$$V = \frac{\int V dt}{\int dt} = \frac{\int_{t-L/2c}^{t+L/2c} V_o \sin \omega t \, dt}{\int_{t-L/2c}^{t+L/2c} dt} \doteq \frac{\sin \frac{\pi L}{\lambda}}{\frac{\pi L}{\lambda}} V_o \sin \omega t \tag{13}$$

where the particle enters the cavity at a time t-L/2c

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and exits at $t + L/2c$, and is at mid-cavity when the field phase is, ideally, $\omega t = \pi/2$. Then, the peak cavity voltage V_o would have an effective value $V = V_oT$ (particle energy gain), where $T = \sin(\pi L/\lambda)/(\pi L/\lambda)$ is a transit time factor which is a measure of the decrease in effectiveness of the gap voltage. Eq (12) may then be put in the quadratic form

$$\frac{V^2}{2R_L T^2} + iV - P_o = 0 \qquad (14)$$

which, solving,

$$V = \sqrt{2P_o R_L T^2 + (iR_L T^2)^2} - iR_L T^2 \qquad (15)$$

The form of the energy gain equation, eq (15), is not useful because it is necessary to know the beam-loaded coupling coefficient (β') to use it, but from eq (7) it is necessary to know the energy gain to compute the coupling coefficient, where the beam resistance is $R = V/i$.

Thus, in the beam-loaded steady state, from eq (7) the coupling coefficient is, noting eq (12),

$$\beta' = \beta\frac{V}{V + R_i} = \beta\,\frac{\sqrt{\frac{8\beta'}{(1+\beta')^2}\frac{P_i T^2}{i^2 R_L} + T^4} - T^2}{\sqrt{\frac{8\beta'}{(1+\beta')^2}\frac{P_i T^2}{i^2 R_L} + T^4} - T^2 + 1} \qquad (16)$$

When the beam-loaded cavity is matched at steady state ($\beta' = 1$), the open circuit coupling coefficient

$$\beta = \frac{\sqrt{\frac{4P_i T^2}{i^2 R_o} + T^4} - T^2 + 1}{\sqrt{\frac{4P_i T^2}{i^2 R_o} + T^4} - T^2} \qquad (17)$$

which is the coupling coefficient to which the cavity must be adjusted for a beam-loaded match.

When eq (17) describes the coupling adjustment the beam-loaded energy gain equation, eq (15), in the steady state is quite complicated owing to the specific dependence of the coupling coefficient on beam-loading current. Observing from eq (7) that

$$\beta' = \frac{\beta(1+\beta)}{1 + \beta + \frac{iR_o}{V}} \qquad (18)$$

one can by re-iteration of eqs (18) and (15) arrive at a beam-loaded energy gain diagram.

But, for the purpose of deriving the precise energy gain equation it is convenient to commence with eq (14) in the expanded form

$$\frac{V}{2R_L T^2} + iV - \frac{4\beta'}{(1+\beta')^2}P_i = 0 \qquad (19)$$

where β' is defined by eq (16), ie.,

$$\beta' = \beta\frac{V}{V + iR_L} \qquad (20)$$

Combining these relations,

$$V^3 + V^2\frac{2iR_oT^2}{(1+\beta)^2}\left(\frac{1}{T^2} + (1+\beta)\right) + V\frac{2R_oT^2}{(1+\beta)^3}\left(\frac{i^2R_o}{2T^2(1+\beta)} + 2i^2R_o - 4\beta P_i\right) + \left(\frac{i^2R_o}{1+\beta} - 4\beta P_i\right)\frac{2iR_oT^2}{(1+\beta)^4} = 0 \qquad (21)$$

At no load eq (21) devolves into

$$V_o = \frac{8\beta}{(1+\beta)^2}P_i R_L T^2 \qquad (22)$$

as expected; the un-anticipated complexity obviously arises from the effects of beam-loading on the couping coefficient. Eq (21)may be factored and the solution is seen to be

$$V = V_o - \frac{\sqrt{2}\, iR_L T}{(1+\beta)} \qquad (23)$$

If the 'cold-test' coupling coefficient β were chosen to produce a match at a specified beam-loading ($i = i_o$ $\beta' = 1$) clearly eq (15) is a correct calculation of the energy gain in the form

$$V = \sqrt{2P_i R_L T^2 + (i_o R_L T^2)^2} - i_o R_L T^2 \qquad (24)$$

but the consequence of beam-loading is poorly predicted by eq (24) when the beam current departs from the value for which the structure was adjusted.

RESONANT PERIODIC STRUCTURES We pass now to the case where a more complicated cavity such as a singly periodic structure is used as the resonant interaction circuit. From an external aspect the periodic waveguide operated resonantly behaves precisely like every other resonant cavity; only details of the beam interaction may differ.

Resonance in such a structure consists of two wave ensembles (the space harmonics) travelling in opposite directions, as a consequence of reflection from the ends of the structure.

Linear homogeneous differential equations whose coefficients are singly periodic functions of the independent variable have solutions which are periodic (6); thus, the solution of the wave equation in a periodic structure may be written

$$E_z = \sum_{-\infty}^{+\infty} a_n e^{j(\omega t - \beta_n z)}$$

$$\beta_n = \beta_o + 2\pi n/p \qquad (25)$$

where p is the periodic length in the structure and a_n are the space harmonic amplitudes. It has been assumed in the above that we are interested in cylindrical coordinates, where the wave equation may be written in the same form for the axial component (3).

If a cavity is formed by placing electric reflectors in the transverse planes of symmetry the propagating modes on the periodic circuit satisfy the boundary conditions of the cavity if the cavity consists of an integral number of half guide-wavelengths; then the standing wave field may be written

$$E_c = \sum a_n e^{j(\omega t - \beta_n z)} + \sum a_n e^{j(\omega t + \beta_n z)} \qquad (26)$$

$$= \sum 2a_n \cos\beta_n z\, e^{j\omega t}$$

In microwave theory shunt impedance is defined as the ratio of RMS voltage developed across a cavity (squared) to the input power. But, when the voltage is meant to be peak voltage, since power is average, a factor of 2 occurs (4). However, in accelerator theory shunt impedance is defined as the ratio of peak electric field intensity (squared) to the power dissipated per unit length (5).

The travelling wave axial electric field amplitude is given by (6)

$$\frac{r}{Q} = \frac{E^2}{\omega w} \tag{27}$$

where w is the linear energy density associated with the travelling wave ensemble,

$$w = \frac{1}{2}\frac{W}{L} \tag{28}$$

that is, is half the total stored energy per unit length. The total stored energy is the product of the cavity Q and power input to the cavity (losses) per radian, which is the fraction of incident power transmitted into the cavity less beam-loading (P_r),

$$w = \frac{P_r Q_L}{2\omega L} \tag{29}$$

Thus,

$$E^2 = \frac{r}{Q}\omega w = \left(\frac{r}{Q}\right)\frac{P_r Q_L}{2L} \tag{30}$$

and the energy gain in transiting the structure

$$V = \int E\,dz = \sqrt{\frac{r}{2}\left(\frac{4\beta'}{(1+\beta')^2}P_i - iV\right)L} \tag{31}$$

Solving this quadratic expression,

$$V = \sqrt{\frac{rL}{2}\,\frac{4\beta'}{(1+\beta')^2}P_i + \left(\frac{irL}{4}\right)^2} - \frac{irL}{4} \tag{32}$$

The energy gain described in eq (32) only applies to the beam current for which the coupling coefficient is β'. When the coupling coefficient value from eq (7),

$$\beta' = \beta\frac{V}{V + irL} \tag{33}$$

is substituted into the above derivation we have the energy gain for any beam current, where β is the 'cold test' adjustment of the coupling coefficient with no beam loading. Performing this algebraic operation,

$$\begin{aligned} &V^3(1+\beta)^2 + V^2\frac{irL}{2}(5+\beta)(1+\beta) \\ &+ V\left((irL)^2(2+\beta) - 2\beta P_i rL\right) \\ &+ \frac{(irL)^3}{2} - 2\beta P_i\, i(rL)^2 = 0 \end{aligned} \tag{34}$$

Eq (34) may be factored

$$\left(V + \frac{irL}{2}\right)\left(V + \frac{irL}{1+\beta}\right)^2 = V_o^2\left(V + irL\right) \tag{35}$$

the solution of which will be seen to be

$$V = V_o - \frac{irL}{\sqrt{2}(1+\beta)} \qquad V_o^2 = \frac{2\beta P_i rL}{(1+\beta)^2} \tag{36}$$

In doubly periodic structures (such as the side coupled LASL cavities and the disc-and-washer) where the RF propagation path does not coincide with the beam path some additional comments are necessary because the accumulative energy gain in this case is somewhat different.

As a very close approximation there is no stored energy in alternate cavities and these are removed from the beam line; the excited cavities contain all the stored energy (W), which at steady state is

$$W = \frac{P_o Q}{\omega} \tag{37}$$

where P_o is total power input to structure.

So that, if the beam line cavity length is d (there are L/d cavities in a specified length of waveguide) the voltage developed across each beam line cavity is given by

$$V_c^2 = \left(\int E\cdot dz\right)^2 = R\left(P_o\frac{d}{L}\right) \tag{38}$$

or, where $r = R/d$ is the shunt impedance per unit length, the voltage across the structure

$$V^2 = P_o rL \tag{39}$$

To derive the steady state, beam-loaded energy gain note that the power available to support the gap voltage is diminished by beam interaction,

$$V^2 = (P_o - iV)rL \tag{40}$$

or, solving,

$$V = \sqrt{P_o rL + \left(\frac{irL}{2}\right)^2} - \frac{irL}{2} \tag{41}$$

But, in this case also, the implied coupling coefficient is only applicable for the specified current. To be precise,

$$P_o = \frac{4\beta'}{(1+\beta')^2}P_i \qquad \beta' = \beta\frac{V}{V+irL} \tag{42}$$

which, being substituted into eq (40),

$$\begin{aligned} &V^3(1+\beta)^2 + V^2 5irL\left(\tfrac{1}{2}+\beta\right) \\ &+ V\left((irL)^2(3+2\beta) - 4\beta P_i rL\right) \\ &+ (irL)^3 - 4\beta(irL)P_i rL = 0 \end{aligned} \tag{43}$$

The solution of a cubic equation, though straightforward, is agonizingly boring, so that it is a welcome observation that eq (43), too, may be factored,

$$(V + irL)\left[\left(V + \frac{irL}{1+\beta}\right)^2 - V_o^2\right] = 0 \tag{44}$$

the solution of which is

$$V = V_o - \frac{irL}{1+\beta} \qquad V_o^2 = \frac{4\beta P_i rL}{(1+\beta)^2} \tag{45}$$

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