IEEE Transactions on Nuclear Science, Vol. NS-28, No. 3, June 1981

ORBIT PROPERTIES FROM THE FIRST INTEGRAL OF THE LORENTZ FORCE

Warren Fenton Stubbins

Department of Physics

University of Cincinnati

Cincinnati, Ohio 45221

The motion of a charged particle in an electric and magnetic field may be studied through the behavior of a vector $\vec{\Delta d}$, analogous to the change in angular momentum, over the particle history. Orbit properties and limits of particle motion may be evaluated. This paper displays the analysis and applies it to a particle in a cylindrically symmetric magnetic field.

Introduction

The properties of orbits in circular accelerators have been examined by various means. The earliest work found that the maximum energy was limited by particle focusing requirements and shortly thereafter the role of betatron oscillations was recognized. Over the years the understanding of the mechanisms for spatial and energy focusing and the relation between them has been advanced and many clever schemes for accelerating particles to very high energies have been proposed and successfully demonstrated.

The Lorentz equation for the force on a charged particle in electrical and magnetic fields is a common foundation for nearly all studies, as it is in this work.

First Integral of the Lorentz Equation

The first integral of the Lorentz equation is found by the vector analysis familiar in the demonstration of Kepler's laws of planetary motion in a central force field. However the Lorentz force is not a central force and the conservation properties of central force systems do not pertain. Thus, we find that the angular momentum, which is conserved in a central force field, depends upon the history of the particle motion in the electric and magnetic fields. It is the examination of the change in the vector formed by the vector product of the radius vector and the velocity vector between endpoints that reveals the dependence of particle motion on the electric and magnetic fields.

The Lorentz force

$$\vec{F}(\vec{r}) = q\ (\vec{E}(\vec{r}) + \vec{v}x\vec{B}(\vec{r})) = M\gamma d^2\vec{r}/dt^2 \qquad (1)$$

governs the motion of a particle with velocity $\vec{v}$ and charge q with a relativistic mass $M\gamma = M(1-v^2/c^2)^{\frac{1}{2}}$ in an electric field $\vec{E}(\vec{r})$ and a magnetic field $\vec{B}(\vec{r})$. The first integral of equation (1) is found by forming the vector product $\vec{r} \times \vec{F}(\vec{r})$ and using the vector identity shown below. The equation becomes

$$M\gamma\vec{r}x\vec{F}(\vec{r}) = M\gamma\vec{r}xd^2r/dt^2 = q$$
$$(\vec{r}xE(\vec{r}) + \vec{r}x\ (dr/dt)\ x\ \vec{B}(\vec{r})\ ). \qquad (2)$$

A change of variable from time to displacement along the orbit path s is made by using d/dt = (ds/dt)(d/ds) = v d/ds and writing p = Mγv where v and p are scalar magnitudes of the particle speed and linear momentum. After rearrangement and noting that

$(d/ds)\ (\vec{r}\ x\ d\vec{r}/ds) = \vec{r}\ x\ d^2\vec{r}/ds^2$, we have

$$(d/ds)\ (\vec{r}\ x\ d\vec{r}/ds) = q[(\vec{r}\ x\ \vec{E}(\vec{r}))/p(s)v(s) + (1/p(s))$$
$$(\vec{r}\ x\ d\vec{r}/ds\ x\ \vec{B}(\vec{r}))] \qquad (3)$$

In writing p(s) and v(s) we are indicating that the magnitude of these scalars are also path dependent when an electrical field is present.

Next we use the vector identity $\vec{A}x\vec{B}x\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$ to restate the last term on the right in equation (3). Equation (3) may be integrated along the orbit path s from endpoint a to endpoint b to give

$$\vec{r}\ x\ d\vec{r}/ds\ \Big|_a^b = q\int_a^b [(\vec{r}\ x\ \vec{E}(\vec{r}))/p(s)v(s)]\ ds$$
$$+\ q\int_a^b [(1/p(s))\ (\vec{r}\cdot\vec{B}(\vec{r})\ d\vec{r}/ds - (\vec{r}\ d\vec{r}/ds)\ \vec{B}(\vec{r})]ds. \qquad (4)$$

Orbits in Axially Symmetric Magnetic Fields

We first consider orbits of particles moving in magnetic fields with no electrical field acting upon them. We set E(r) = 0 and take p(s) = p outside the integral. We adopt a cylindrical coordinate frame and examine the orbit properties which are those of orbits in a conventional cyclotron. This enables the details of the method to be displayed and the results compared to well known orbit properties.

The left hand side of equation (4) is the difference between the vector $\vec{d}(s) = \vec{r}\ x\ d\vec{r}/ds$ evaluated at the endpoint b and the endpoint a. We define

$$\Delta\vec{d} = \vec{d}(b) - \vec{d}(a). \qquad (5)$$

The component equations for each of the three components of $\Delta\vec{d}$ from equation (4) with E(r) = 0 are

$$\Delta d_\rho = -Z\rho d\emptyset/ds\ \Big|_a^b = (q/p)\int_a^b [Z\ B_z\ d\rho/ds - B_\rho Z\ dZ/ds]ds \qquad (6)$$

$$d_\emptyset = (Z\ d\rho/ds - \rho dZ/ds)\ \Big|_a^b = (q/p)\int_a^b [\rho^2 B_\rho\ d\emptyset/ds + Z\rho B_z\ d\emptyset/ds]\ ds \qquad (7)$$
$$-\int_a^b[(\rho d\rho/ds + Z\ dZ/ds)\ B_\emptyset]\ ds$$

$$\Delta d_z = \rho^2 d\emptyset/ds\ \Big|_a^b = (q/p)\int_a^b[\rho B_\rho dZ/ds - B_z\ \rho d\rho/ds\ ]\ ds \qquad (8)$$

The quantity d∅/ds may be found from the metric in in cylindrical coordinates, $ds^2 = d\rho^2 + \rho^2 d\emptyset^2 + dZ^2$, and is

0018-9499/81/0600-2615$00.75©1981 IEEE

$$d\phi/ds = (1/\rho)\ [1 - (d\rho/ds)^2 - (dZ/ds)^2\ ]^{1/2} \tag{9}$$

The above equations are simplified when the magnetic field is restricted to one without asimuthal variations, $B_\phi = 0$, corresponding to a convential cyclotron. We examine particle motion in the midplane of the axially symmetric magnetic field with the further simplification that the axial motion is zero, $Z = 0$. Under these two restrictions the component equations become

$$\Delta d\rho = 0,$$

$$\Delta d_\phi = 0 \quad \text{and} \tag{10}$$

$$\Delta d_z = \rho(1 - (d\rho/ds)^2\ )^{1/2}\ \Big|_a^b = -(q/p)\ \int_a^b\ B_z\ \rho d\rho.$$

The radial motion in the midplane is conveniently described as the departure from a closed circular orbit, called the reference orbit, of radius R. Thus we write

$$\rho = R + X. \tag{11}$$

In the simplist approximation the departure from the reference orbit is a sinusoid so we set $X = X_o \sin Ks$ and we have

$$\rho = R + X_o \sin Ks. \tag{11a}$$

Along the reference orbit concentric with the center of the cyclotron and on the midplane the centripetal and magnetic forces are equal, thus

$$M\gamma v^2/\ R = q\ B_o v \tag{12}$$

which yields

$$1/R = q\ B_o/p. \tag{12a}$$

B_o is the magnetic induction B_z at the radius R and on the midplane, and $p = M\gamma v$ is the particle momentum.

A usual description of the magnetic field in a conventional cyclotron in the neighborhood of a closed orbit is

$$B_z(r) = B_o(\rho/R)^{-n}\ . \tag{13}$$

The magnetic field index n, defined by equation (13), measures the rate of magnetic field fall off with radius.

$$n = -\ (R/B_o)(dB/dr)\Big|_{r=R}. \tag{14}$$

Radial Oscillation Frequency

The frequency of the radial oscillations about the reference orbit can be found by using equations (10), (11a) and (13) and simple endpoints for motion in the midplane. We chose the path origin, $s = a = 0$, as the lower endpoint which yields $\rho(a) = \rho(0) = R$. The origin is the point at which the particle is crossing the closed reference orbit. The other endpoint b is chosen to allow $Ks = Kb = \pi/2$ which is the point of maximum departure from the reference orbit. With these endpoints equation (10) becomes

$$\Delta d_z = \rho(1-(d\rho/ds)^2)^{1/2}\ \Big|_a^b = -(q/p)\quad \int_a^b\ (\rho/R)^{-n}\rho d\rho$$

$$= (1/(2-n)R^{1-n})\rho^{2-n}\ \Big|_a^b \tag{15}$$

where we have used equation (12) and the fact that for positively charged particles B_z is directed along the negative z-axis to have closed orbits. This disposes of the negative sign in equation (15).

The evaluation of both sides of equation (15) is done by expanding ρ and $d\rho/ds$ with the appropriate binomial series. When the endpoints are applied we have

$$R\ [1 - (KX_o)^2/2 - (KX_o)^4/8 - ..\ ] - (R + X_o) \simeq \tag{16}$$

$$- X_o - (1-n)X_o^2/2R + n(1-n)X_o^3/6R^2 + . .$$

This is solved as a quadratic in K^2 to yield

$$K^2 = [(1-n)/R^2][1-(n/3)(X_o/R) - ((1-n)/8)(X_o/R)^2 + . .]. \tag{17}$$

We write $K = k/R$ in accord with the dimensions of Ks where k is a dimensionless parameter. The lowest order term in equation (17) shows

$$k = (1-n)^{1/2}$$

which is the well known first order radial oscillation frequency term in a convential cyclotron. The radial oscillation frequency depends upon oscillation amplitude as shown by equation (17).

Orbits in a Flat Magnetic Field

The motion of charged particles in a uniform magnetic field, field index $n = 0$, is resolved into a linear component of velocity along the field lines and a circular orbit around the field lines. This example illustrates the significance of radial component of $\vec{\Delta d}$.

For a flat field $B_z = B_o$ and with the condition that $dZ/ds = \text{constant} \neq 0$ we have from equation (8)

$$\Delta d_z = \rho[1 -(d\rho/ds)^2 - (dZ/ds)^2]^{1/2}\ \Big|_a^b = -(q/p)\ \int_a^b B_z\rho d\rho \tag{18}$$

$$= [1 - (dZ/ds)^2]^{1/2}\ [1 -(d\rho/ds)^2/(1 -(dZ/ds)^2]^{1/2}$$

Now when we use $\rho = R + X_o \sin Ks$ and evaluate K as above we find that $K^2 = 1/R^2$ or $k = 1$ indicating that there is no precession and no restoring force for any initial orbit orientation. However in obtaining this result it is necessary to note that qB_o/p is not simply $1/R$ because the momentum p has a component in the axial direction. The relation we use $qB_o/p = (1 - (dZ/ds)^2)^{1/2}/R$ comes from the resolution of the momentum into an axial and a radial component.

By choosing the z-axis to be the guiding center, the center of the circular orbit of radius R, the radial component $\vec{\Delta d}$ given by equation (6) may be evaluated. We use $X_o = 0$ and find

$$\Delta d_\rho = -Z\,[1 - (dZ/ds)^2]^{\frac{1}{2}}\Big|_a^b = -(qB_o/p)\,Z \int_a^b \rho\, d\rho \quad (19)$$

which is used with endpoints on the z-axis Z(a) and Z(b) to get

$$\Delta d_\rho = [-Z(b) + Z(a)]\,[1 - (dZ/ds)^2]^{\frac{1}{2}} \quad (19a)$$

As shown in Figure 1 the radial component of $\vec{\Delta}d = \vec{r} \times d\vec{r}/ds \Big|_a^b$ grows linearly as Z(b) - Z(a) increases. The vector component Δd_ρ is opposite in direction to the unit radial vector.

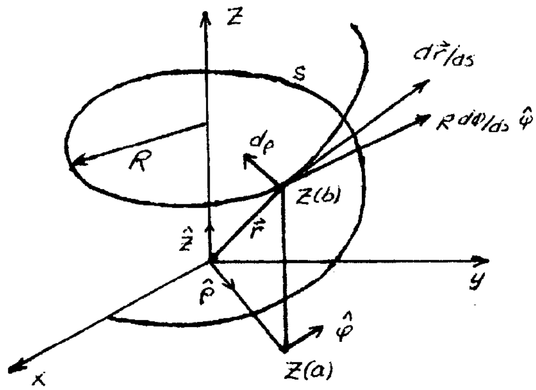

Figure 1 Particle Motion in a Flat Magnetic Field

General Properties of Component Equations

Equations (6), (7) and (8) may be seen with the aid of the two examples as the key for understanding orbit excursions in axially symmetric magnetic field. Δd_ρ, equation (6) measures the axial departure from the midplane. The limits set by dee aperatures or other restrictions on axial motion may be related to the magnitude of Δd_ρ. The suitability of initial amplitudes of axial and radial motions and the magnetic field parameters may be determined.

Δd_z, equation (8), is the difference between two rather large terms each approximately R in magnitude. The comparison of equations (6) and (8) for Δd_ρ and Δd_z, respectively, reveals the symmetry between radial and axial motions.

Δd_ρ, equation (7), emphasizes the coupling between radial and axial oscillations. The last term in equation (7) has the nature of a driving force arising from azimuthal variations in the magnetic field and provides a means of evaluating the effects of magnetic asymmetry.

Contribution of an Electric Field

The use of this analysis with an electric field is complicated by the dependence of both p(s) and v(s) upon the change in particle energy along the path. This is recognized in equation (4) where p(s) and v(s) are within the integral. The contribution to $\vec{\Delta}d$ by the electric field may be seen in the component equations for the portion of equation (4) containing the electric field $\vec{E}(\vec{r})$. These are

$$\Delta d_\rho = -q \int_a^b [(Z\,E_\phi)/p(s)v(s)]\,ds \quad (20)$$

$$\Delta d_\phi = q \int_a^b [(Z\,E_\rho - \rho E_z)/p(s)v(s)]\,ds \quad (21)$$

$$\Delta d_z = q \int_a^b [(\rho E_\rho)/p(s)v(s)]\,ds. \quad (22)$$

When the accelerating field is applied over a limited portion of the path corresponding to a dee gap crossing the contribution of the electric field may be evaluated as a step change. This is a very useful and frequently valid simplification. If this is done then the new magnitudes of p(s) and v(s) are considered as a new scaling for the next portion of the orbit.

Conclusion

The vector $\vec{\Delta}d$ from the first integral of the Lorentz equation may be analytically (and computationally) studied to evaluate orbit properties arising from primary and perturbative fields in which the particle moves.