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PARTICLE DYNAMICS IN THE ELECTRON LINEAR ACCELERATOR

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This report is a discussion of the equation of motion in the electron linac and conditions for solution. For one special case, the constant gradient, velocity of light section, a solution is obtained.

Given equations of the form

$$\frac{d\gamma}{d\xi} = f(\delta) \qquad \frac{d\delta}{d\xi} = F(\gamma) \tag{1}$$

which correspond to the one-dimensional equations of motion for a particle interacting with a travelling wave, as occurs in the electron linac, the so-called phase trajectory solution

$$F(\gamma)\,d\gamma = f(\delta)\,d\delta \tag{2}$$

can be gotten, which then provides the γ-δ relation necessary to solve eq (1). If eqs (1) are more complicated, separability of variables is the only additional requirement. While this problem is somewhat conveniently solved by computer using a Runge-Kutta, Milne or some other algorithm, an explicit solution can be obtained, at least in the case where there are no ξ-dependent parameters.

In the particle-wave interaction in the accelerator, the axial equation of motion describes the energy gain of the particle,

$$\frac{d}{dt}\left(\gamma\frac{dz}{dt}\right) = -\frac{e}{m_0}E_z \tag{3}$$

Taking the indicated derivative, or as is obvious from energy considerations,

$$\frac{d\gamma}{dz} = -\frac{e}{m_0c^2}E_z \tag{4}$$

For the purpose of normalizing this equation for computer programming we will use the free-space wavelength as an independent variable, $\xi = z/\lambda$, and normalize the electric field intensity (as energy gain per free-space wavelength), $\alpha = eE\lambda/m_0c^2$. Thus, if δ is the phase of the particle with respect to the field zero

$$\frac{d\gamma}{d\xi} = -\alpha\sin\delta \tag{5}$$

The force on a particle produces an acceleration which will, in general, vary the motion and phase of the particle with respect to the wave. This displacement is

$$d\left(\frac{\delta}{2\pi}\lambda_g\right) = (v_e - v_w)\,dt \tag{6}$$

The time differential may be expressed in terms of the distance traversed by the electron (in free-space wavelengths)

$$d\delta = \frac{2\pi}{\lambda_g}(v_e - v_w)\frac{dz}{v_e} = 2\pi\left(\frac{1}{\beta_w} - \frac{1}{\beta_e}\right)d\xi \tag{7}$$

where $\beta_w = v_w/c$ and $\beta_e = v_e/c = \sqrt{\gamma^2-1}/\gamma$. Eqs (5) and (7) specify the longitudinal motion of the electron in terms of properties of the wave in the guide $\alpha(\xi)$ and $\beta_w(\xi)$, ignoring other components of the field and space charge effects arising from other particles in the 'bunch' (1). These equations are coupled and, usually are solved by numerical integration: however, in certain cases an analytic solution can be obtained.

In particular, if $\alpha(\xi)$ and $\beta_w(\xi)$ are constant (e.g., constant gradient, velocity-of-light sections) equations (5) and (7) can be solved (eliminating $d\xi$)

$$2\pi\left(\frac{1}{\beta_w} - \frac{\gamma}{\sqrt{\gamma^2-1}}\right)d\gamma = -\alpha\sin\delta\,d\delta \tag{8}$$

which, integrating

$$\left(\frac{\gamma}{\beta_w} - \sqrt{\gamma^2-1}\right) - \left(\frac{\gamma_0}{\beta_w} - \sqrt{\gamma_0^2-1}\right) = \frac{\alpha}{2\pi}\left(\cos\delta - \cos\delta_0\right) \tag{9}$$

Some observations on the trajectories of particles is immediately evident. When $\beta_w < 1$ γ may be more or less than γ_0, because δ is cyclic, varying between $\pm\delta_0$. The motion, when relativistic changes of the mass of the particle may be neglected, is analogous to that of the pendulum (ie, is isochronous), but generally as in pendulous motion, is an elliptic function (ie, non-isochronous, or dependent on phase amplitude). Energy oscillation (γ) is symmetrical about the injection energy (γ_0), for small amplitudes.

On the other hand, when $\beta_w = 1$ the nature of the trajectories change because $\gamma - \sqrt{\gamma^2-1} \to 0$. For particles which are captured (bound to the wave) γ increases without limit as ξ increase and $\gamma - \sqrt{\gamma^2-1} = 0$, so that

$$\cos\delta_\infty = \cos\delta_0 - \frac{2\pi}{\alpha}\left(\gamma_0 - \sqrt{\gamma_0^2-1}\right) \tag{10}$$

δ_∞ is then the 'synchronous' or 'asymptotic' phase ultimately approached by the particle.

This same observation may be shown in another way. From eq (7) we may write

$$\frac{d\delta}{d\xi} = 2\pi\left(\frac{1}{\beta_w} - \frac{\gamma}{\sqrt{\gamma^2-1}}\right) \tag{11}$$

which for bound particles becomes, since $\gamma/\sqrt{\gamma^2-1} \to 1$

$$d\delta = 2\pi\left(\frac{1-\beta_w}{\beta_w}\right)d\xi \tag{12}$$

from which we may conclude that in the velocity-of-light sections highly energetic particles do not change phase, and moreover, that there is no phase compression (bunching) in the velocity-of-light sections. The phase width of a highly relativistic bunch remains unchanged in a high energy accelerator, disregarding space-charge effects and errors in waveguide tuning. An extensive discussion of the beam phase errors owing to RF defects has been given by Belbeoch. (2)

With the phase trajectory solution, eq (9), we are now able to solve eq (5)

$$\frac{d\gamma}{d\xi} = -\alpha\sin\delta = \pm\alpha\sqrt{1-\cos^2\delta} \tag{13}$$

From eq (13) and letting

$$\eta = 1 - \left(\cos\delta_0 - \frac{\gamma_0 - \sqrt{\gamma_0^2-1}}{\alpha/2\pi}\right)$$
$$\zeta = \frac{1}{\alpha/2\pi}\left(\cos\delta_0 - \frac{\gamma_0 - \sqrt{\gamma_0^2-1}}{\alpha/2\pi}\right), \tag{14}$$
$$1 - \cos^2\delta = \eta - \frac{\zeta}{\gamma}$$

where we have set $\beta_w = 1$ and expanded $\sqrt{\gamma^2-1}$ by the binomial theorem. Eq (13) is now an integrable form;

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$$\frac{\gamma\, d\gamma}{\sqrt{\eta\gamma^2-\xi\gamma}} = \alpha\eta\, d\xi \tag{15}$$

the solution of which is

$$\sqrt{\eta\gamma^2-\xi\gamma} - \sqrt{\eta\gamma_0^2-\xi\gamma_0} + \frac{\xi}{\sqrt{\eta}} \ln\frac{\sqrt{\gamma}+\sqrt{\gamma-\frac{\xi}{\eta}}}{\sqrt{\gamma_0}+\sqrt{\gamma_0-\frac{\xi}{\eta}}} = \alpha\eta\xi \tag{16}$$

It will not escape the reader's notice that eq. (9) is quadratic in γ , the solution of which is

$$\gamma = \frac{-x+\sqrt{\beta_w^2 x^2+4(1-1/\beta_w^2)}}{2(1-1/\beta_w^2)} \tag{17}$$

$$x = \frac{\alpha}{\beta_w \pi}\left(\cos\delta - \cos\delta_0\right) + \frac{2}{\beta_w}\left(\frac{\gamma_0}{\beta_w} - \sqrt{\gamma_0^2-1}\right)$$

Again, one can see the profound change in the nature of the solution when $\beta_w = 1$.

From eq. (10) the capture angle of the velocity-of-light section may be calculated, since the magnitude of cos δ_{∞} must be less than unity. For example, for $\gamma_0 = 1.157$ (80 KV) and $\alpha = 4$ intrant particles $95.5^\circ < \delta_0 < 95.5^\circ$ provide a 'real' solution; that is, about 191° of input phase will be captured into the accelerating half cycle (but with a poor spectrum).

Another calculation of the same mathematical kind, though physically very different is that of the trajectory of a particle (electron) transiting a cavity). The problem has been proposed by the LASL PHERMEX group and Prof. Septier, CNRS, Orsay (3).

The trajectory of a particle traversing the cavity, derived from a simplified version of the equations of motion (neglecting other field components), is

$$\frac{d}{dt}\left(\gamma\frac{dz}{dt}\right) = \frac{eE_0}{m_0}\sin(\omega t+\varphi_0) \tag{18}$$

where E_0 is the axial field intensity (assumed constant here) and φ_0 is the intrant phase of the particle. Thus, integrating,

$$\gamma\left(\frac{1}{c}\frac{dz}{dt}\right) - \gamma_0\left(\frac{1}{c}\frac{dz}{dt}\right)_0 = \frac{eE_0\lambda}{m_0c^2 2\pi}\left(\cos\varphi_0 - \cos(\omega t+\varphi_0)\right) \tag{19}$$

where the normalized intrant energy is γ_0. Noting that

$$\frac{1}{c}\frac{dz}{dt} = \frac{\sqrt{\gamma^2-1}}{\gamma}$$

and setting $\varphi = \omega t + \varphi_0$, $\alpha = eE_0\lambda/m_0c^2 2\pi$

$$\frac{dz}{d\varphi} = \frac{\lambda}{2\pi}\,\frac{\alpha(\cos\varphi_0-\cos\varphi)+\sqrt{\gamma_0^2-1}}{\sqrt{\left(\alpha(\cos\varphi_0-\cos\varphi)+\sqrt{\gamma_0^2-1}\right)^2+1}} \tag{20}$$

Re-integration of this expression seems to be formidable and one would likely proceed by numerical integration, carried out by starting at an intrant phase φ_0 and energy γ_0 and choosing an interval of integration $\Delta\varphi$; thus, φ will be $\varphi_0+\Delta\varphi$ and a value of Δz calculated. Then, $\varphi_0+\Delta\varphi$ becomes the intrant phase for the next interval and γ (calculated from eq. (21)),

$$\gamma = \sqrt{\left(\alpha(\cos\varphi_0-\cos\varphi)+\sqrt{\gamma_0^2-1}\right)^2+1} \tag{21}$$

the intrant energy for the interval. The process is continued until the sum the distances Δz is the length of the cavity ($\sum\Delta z = L$), where the exit phase and energy are then known. Obviously eq (21) provides a consistency check on the whole process.

But, note that eq. (20) can be expanded by the binomial theorem and integrated to any degree of precision using the recurrence integrals

$$\int\frac{d\varphi}{(a+b\cos\varphi)^n} = \frac{b\sin\varphi}{(n-1)(a^2-b^2)(a+b\cos\varphi)^{n-1}} + \frac{2n-3}{(n-1)(a^2-b^2)}\int\frac{d\varphi}{(a+b\cos\varphi)^{n-1}} - \frac{n-2}{(n-1)(a^2-b^2)}\int\frac{d\varphi}{(a+b\cos\varphi)^{n-2}}$$

$$\int\frac{d\varphi}{a+b\cos\varphi} = \frac{2}{\sqrt{a^2-b^2}}\arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\varphi}{2}\right) \tag{22}$$

Thus, to two terms, eq. (20) becomes

$$\frac{2\pi}{\lambda}dz = \left(1-\frac{1}{2}\,\frac{1}{\left(\alpha\cos\varphi_0+\sqrt{\gamma_0^2-1}-\alpha\cos\varphi\right)^2}\right)d\varphi \tag{23}$$

which, integrating,

$$\frac{2\pi L}{\lambda} = \Bigg[\varphi + \frac{\alpha\sin\varphi}{2\left(\left(\alpha\cos\varphi_0+\sqrt{\gamma_0^2-1}\right)^2-\alpha^2\right)\left(\alpha\cos\varphi_0+\sqrt{\gamma_0^2-1}-\alpha\cos\varphi\right)} - \frac{\alpha\cos\varphi_0+\sqrt{\gamma_0^2-1}}{\left(\left(\alpha\cos\varphi_0+\sqrt{\gamma_0^2-1}\right)^2-\alpha^2\right)^{3/2}}\tan^{-1}\left(\sqrt{\frac{\alpha\cos\varphi_0+\sqrt{\gamma_0^2-1}+\alpha}{\alpha\cos\varphi_0+\sqrt{\gamma_0^2-1}-\alpha}}\tan\frac{\varphi}{2}\right)\Bigg]_{\varphi_0}^{\varphi} \tag{24}$$

The implicit character of φ in the above expression greatly detracts from its usefulness; the result in greater detail is an arithmetical jungle and one would resort to machine for computational assistance (which was the original situation), and this latter program is, moreover, much more complicated. That this complication stems from relativistic kinetics may be seen from the classical solution:

$$\frac{d(mv)}{dt} = eE_0\sin(\omega t+\varphi_0) \tag{25}$$

Integrating, with boundary conditions t = 0, $\varphi = \varphi_0$, v = v_0;

$$v = \frac{eE_0}{m\omega}\left(\cos(\omega t+\varphi_0) - \cos\varphi_0\right) + v_0 \tag{26}$$

Re-integrating, with t = 0, z = 0 and putting $\varphi = \omega t + \varphi_0$, $E_0 = V_0/L$;

$$z = \frac{eV_0}{Lm\omega^2}\left(\sin\varphi - \sin\varphi_0 - (\varphi-\varphi_0)\cos\varphi_0\right) + v_0 t \tag{27}$$

Setting z = L and $\alpha = (eV_0/mc^2)(\lambda/2\pi L)^2$

$$\sin\varphi - \varphi\left(\cos\varphi_0 - \frac{v_0\lambda}{c\,2\pi}\right) = \sin\varphi_0 - \varphi_0\left(\cos\varphi_0 - \frac{v_0\lambda}{c\,2\pi}\right) + \frac{1}{\alpha} \tag{28}$$

Eq. (28), like eq. (24), are forms of Kepler's transcendental equation. (4) It will be evident that either eq. (28) or eq. (24) is also the solution of the single gap modulation problem (klystron cavities, linac pre-buncher, etc.) where V_0 is the gap voltage and $(v_0/c)^2 = 2\ eV/mc^2$ relates the gun voltage to the gap input velocity, non-relativistically.

Returning now to the previous discussion of eqs. (5) and (7), it seems incredible that one cannot solve this pair of equations in general, but it appears that if $\alpha(\xi)$ and $\beta(\xi)$ are not constant a 'phase trajectory'[5] cannot be obtained and that is the basic problem.

The area of principal interest in the solution of[6] the trajectory equations is in the design of bunchers. As observed earlier, when $\beta_w < 1$ electrons oscillate around a hypothetical orbit and this is so even if the phase velocity is continually changing. In principle we can specify the orbit of at least one electron trajectory in the buncher. For example, if we want one particle (also called the synchronous particle) to remain stationary with respect to the wave,

$$\frac{d\delta}{d\xi} = 2\pi\left(\frac{1}{\beta_w(\xi)} - \frac{\gamma_s}{\sqrt{\gamma_s^2-1}}\right) = 0 \tag{29}$$

$$\text{ie.}\quad \gamma_s = \frac{1}{\sqrt{1-\beta_w^2}}$$

which is that the field strength in the structure be given by

$$\frac{d\gamma_s}{d\xi} = \frac{\beta_w(\xi)}{(1-\beta_w^2(\xi))^{3/2}}\frac{d\beta_w(\xi)}{d\xi} = -\alpha(\xi)\sin\delta_s \tag{30}$$

Clearly, one can choose a more complicated phase history for the particle with intrant phase δ_0 by specifying $d\delta/d\xi$, thereby stating the function $\delta(\xi)$ for one elctron. Again, in this case

$$\gamma(\xi) = \frac{1}{\sqrt{1-\dfrac{1}{\left(\dfrac{1}{\beta_w(\xi)}-\dfrac{1}{2\pi}\dfrac{d\delta}{d\xi}\right)^2}}} \tag{31}$$

and, as before,

$$\frac{F^2(2-F^2)}{(F^2-1)^{3/2}}\frac{dF}{d\xi} = \alpha(\xi)\sin\delta(\xi) \tag{32}$$

$$F = \left(\frac{1}{\beta_w(\xi)} - \frac{1}{2\pi}\frac{d\delta}{d\xi}\right)$$

from which the required field strength may be determined. The calculations involved are tedious and all other trajectories must still be found by numerical integration of eqs. (5) and (7). The value of the observation of eqs. (31) and (32) lies in being able to specify precisely the trajectory of at least one particle. An example of this process cannot be presented from the necessity of briefness. The practicality of the above technique of finding useful functions $\alpha(\xi)$ and $\beta_w(\xi)$ is only limited by the realizability of the microwave structure, as described in ref. 6.

After having found an acceptable set of functions of α and β_w for individual orbits, it will be objected that the microwave design will require a knowledge of the effect of beam-loading upon the power flux and, consequently, on the electric field, the which can be supplied by supposing the total bunched current as oscillating about the 'synchronous' orbit and on the average gaining energy at the same rate as that hypothetical particle. A more serious consideration is the effect of other particles in the bunch on the trajectory of a member of it. A technique of attack upon this problem has been provided by several investigators (7), using the Tien model of the beam(8). This approach is to consider the beam as a series of charged discs, ascribing all the beam charge to N discs per wavelength. The potential of such a disc in an enclosure can be obtained. The force on any disc may then be determined by summing over all other discs which added to the applied field permits the motion of the disc to be calculated. Because the discs are not rigid bodies, some further sophistication in describing charge elements has been attempted. (9)

Progress in the solution of this sort of non-linear differential equation has not achieved any substantial results in the last thirty years, since E. L. Chu presented a full account of what was then tractable (10). That is not to suggest that no effort has been made; Ref. (11), for example, is a compilation of 53 papers devoted to a discussion of similar and the more generalized problem from considerations of the memoirs of H. Poincaré, A. Liapunov, G. Birkhoff and S. Lefshetz.

From a purely mathematical view-point Poincaré generalized the dynamical problem in celestial mechanics to

$$\frac{d\gamma}{d\xi} = P(\gamma,\delta) \qquad \frac{d\delta}{d\xi} = Q(\gamma,\delta) \tag{33}$$

where P and Q are real polynomials. The so-called 'allure' or 'phase portrait' was understood by Poincaré (1899) to be the essence of the solution;

$$\frac{d\gamma}{d\delta} = \frac{P(\gamma,\delta)}{Q(\gamma,\delta)} \tag{34}$$

where P and Q, though arbitrary, are 'relatively prime'. Poincaré also noted that a full description of the phase trajectory depended upon determination certain critical attributes; nodes, saddle points, foci, limit cycles, separatrices, indices, etc. As will be readily observed, Poincaré had in mind (in drawing attention to the phase portrait) that the third variable is momentarily removed from consideration, so that then some process, such as separation of variables, will permit an ultimate solution. Ignoring special cases, that is not the situation here. We are faced with the problem

$$\frac{d\gamma}{d\xi} = P(\xi,\delta) \qquad \frac{d\delta}{d\xi} = Q(\xi,\gamma) \tag{35}$$

so that the phase portrait cannot be obtained in so simple a manner. It appears necessary to accept a higher order differential equation to be solved, where with substitutions one variable can be eliminated. Given that

$$\frac{d\gamma}{d\xi} = -\alpha\sin\delta$$
$$\frac{d\delta}{d\xi} = 2\pi\left(\frac{1}{\beta_w} - \frac{\gamma}{\sqrt{\gamma^2-1}}\right) \tag{36}$$

then

$$\frac{d^2\gamma}{d\xi^2} = -\alpha\cos\delta\frac{d\delta}{d\xi} - \frac{d\alpha}{d\xi}\sin\delta \tag{37}$$
$$= -2\pi\alpha\sqrt{1-\left(\frac{1}{\alpha}\frac{d\gamma}{d\xi}\right)^2}\left(\frac{1}{\beta_w} - \frac{\gamma}{\sqrt{\gamma^2-1}}\right) + \frac{1}{\alpha}\frac{d\alpha}{d\xi}\frac{d\gamma}{d\xi}$$

Now, $\alpha(\xi)$ and $\beta_w(\xi)$ are, of course, known so that eq. (37) may, in principle, be solved, although non-linear differential equations tax the ingenuity and insight of any mathematician. Although the final solution of eqs. (36) is not of great importance to designers of bunchers it is of heuristic and practical interest; it is implausible that an extended knowledge of beam dynamics would result in poorer buncher designs.

(1) These equations have also been derived by J. C. Slater from considerations of the Hamiltonian, Rev. Mod. Phys. 20,473 (1948)

(2) R. Belbeoch, A Study of Beam Phase Coherence, Project M Int. Mem. Rep. M-279 (SLAC) Stanford U.(1961)

(3) D. Venable, ed. PHERMEX, LASL Rep. No. LA-3241 (1967)

M. Boussoukaya et A. Septier, Particle Accelerators 2 (4), 305 (Oct., 1971)

(4) See F. R. Moulton, Celestial Mechanics, NY, 1914; The Bulletin Astronomique, Jan. 1900, contains references to 123 papers on the solution of this equation. The treatment in Watson's Bessel Functions, Camb. U.P. (1922) p. 553 is a complete analytical solution.

(5) A mathematical discussion of this type of problem is given by N. Minorsky Intro. to Non-linear Mechanics, Ann Arbor, Mich. (1947)

(6) W. J. Gallagher, IEEE Trans. Nuc. Sci. NS-24(3), 1716 (1977) which also contains references to the principal engineering discussions on buncher design.

(7) M. Mac Gregor and C. B. Williams, IEEE Trans. Nuc. Sci. NS-14,581 (1967)

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(8) P. K. Tien et al., Proc. IRE 43,260 (1955)

(9) J. Haimison et al. IEEE Trans. Nuc. Sci. NS-14, 586 (1967)

(10) E. L. Chu, The Theory of the Linear Electron Accelerator, ML Report 140 Stanford Univ. (1951)

(11) Differential Equations and Dynamical Systems, jt. ed. J. Hale, Academic Press (1965)