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# Space Charge Expansion of Single Packets of Charge

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## Introductory Comments

The history of particle electrodynamics has followed from the disquisitions of H. A. Lorentz, M. Abraham and A. Föppl (1). These were, of course, non-relativistic electron theories, that is, attempts to explain the electromagnetic properties of matter as owing to the motion of discrete electrical charges. Of course, J. C. Maxwell was not aware of the existence of electron and his theory (which is inadequate for the problem) only addressed itself to an infinitesimally attenuable electric fluid. By 1912 L. Page at Yale formulated the vector form of electrodynamics as used today (2). But until recently some physicists did not accept certain implications in Page's formulation; particularly, that the force on a charge did not depend on the motion of the charge but only upon the motion of the source causing the field (3). As late as 1947 other serious doubts about the formulation of electrodynamics were raised among electrical engineers (4). With time most of these objections have dissipated away, and it is generally agreed in 'classical' electrodynamics that the electron be the Lorentz electron, that the field caused by an electron in motion does not affect the particle causing the field and that the force on an electron in a field does not depend on its own motion (other than that the 'force' equation is Lorentz's).

Following the development of the theory of relativity it was necessary to transform electromagnet fields between frames of reference in relative motion so as to preserve the form of Maxwell's equations. If in a system at 'rest' there is an electric and magnetic field E and B we may, using the Lorentz, calculate the fields E' and B' in a system in 'motion' with a velocity $\bar{v}$. The components parallel to v remain unchanged, but components perpendicular to $\bar{v}$ are transformed according to the rule:

$$\bar{E}' = \frac{\bar{E} + \bar{v}/c \times \bar{B}}{\sqrt{1-(v/c)^2}} \qquad \bar{B}' = \frac{\bar{B} - \bar{v}/c \times \bar{E}}{\sqrt{1-(v/c)^2}} \tag{1}$$

The remarks above are concerned with the transformation of fields for a moving observer. A more direct consideration of the electrodynamic problem (that is, the force between moving charges) leads to the following.

The field at the origin of the coordinate system S as a consequence of the charge $e_1$ $(x_1, y_1, z_1)$ moving with a velocity $v_1$ in S parallel to the x-axis is

$$E_x = -(1-\beta^2)\frac{e_1 x_1}{4\pi\epsilon_0 S_1^3} \qquad H_x = 0 \tag{2}$$

$$E_y = -(1-\beta^2)\frac{e_1 y_1}{4\pi\epsilon_0 S_1^3} \qquad H_y = \beta_1(1-\beta_1^2)\frac{e_1 z_1}{4\pi\epsilon_0 S_1^3}$$

$$E_z = -(1-\beta_1^2)\frac{e_1 z_1}{4\pi\epsilon_0 S_1^3} \qquad H_z = -\beta_1(1-\beta_1^2)\frac{e_1 y_1}{4\pi\epsilon_0 S_1^3}$$

$$\beta_1 = v/c$$

$$S_1^2 = x_1^2 + (1-\beta_1^2)(y_1^2 + z_1^2)$$

The units are MKS and the coordinate system is chosen so that a second particle $e_2(x_2, y_2, z_2)$ with velocity $v_2$ in the frame of reference S' may be oriented such that $v_2$ is parallel to the x'-axis and the origin of the coordinates coincide.

In general the force on $e_2$ may be calculated from the Lorentz 'force equation':

$$\begin{aligned} F_x &= e_2 E_x + \frac{e_2}{c}(v_y H_z - v_z H_y) \\ F_y &= e_2 E_y + \frac{e_2}{c}(v_z H_x - v_x H_z) \\ F_z &= e_2 E_z + \frac{e_2}{c}(v_x H_y - v_y H_x) \end{aligned} \tag{3}$$

where it is apparent that the use of vector notation, which merely transcends the details of the coordinate system, results in a more elegant form but for computational purposes one must write the result in some (orthogonal) coordinate system.

In the particular case, where the motion with velocity $v_2$ of $e_2$ is parallel to $e_1$, (as we suppose occurs in accelerator waveguides) i.e. the x and x' axes coincide, eqs(3) become:

$$\begin{aligned} F_x &= e_2 e_1 (1-\beta_1^2)\frac{x_1}{4\pi\epsilon_0 S_1^3} \\ F_y &= -e_2 e_1 (1-\beta_1^2)(1-\beta_1\beta_2)\frac{y_1}{4\pi\epsilon_0 S_1^3} \\ F_z &= -e_2 e_1 (1-\beta_1^2)(1-\beta_1\beta_2)\frac{z_1}{4\pi\epsilon_0 S_1^3} \end{aligned} \tag{4}$$

Of course, to calculate the motion of $e_2$ we must use relativistic kinematics

$$\begin{aligned} F_x &= \frac{d}{dt}(\gamma_2 m_0 v_x) = \frac{m_0}{\sqrt{1-\beta_2^2}} \frac{d v_{z2}}{dt} \\ F_y &= \frac{d}{dt}(\gamma_2 m_0 v_y) = \frac{m_{02}}{\sqrt{1-\beta_2^2}} \frac{d v_{y2}}{dt} \\ F_z &= \frac{d}{dt}(\gamma_2 m_0 v_z) = \frac{m_{02}}{\sqrt{1-\beta_2^2}} \frac{d v_{z2}}{dt} \end{aligned} \tag{5}$$

It should be noted that the procedure is to perform all calculations in the laboratory (or 'rest'system). It is also obvious that the force between two charged particles in relative motion is not along the line between them.

We are also especially interested in the case where two like particles move abreast with equal velocities (suppose along the z-axis, separated a distance y); then from eqs(4) and (5), since z = x = 0 and $\beta_1 = \beta_2$,

$$\frac{d^2y}{dt^2} = \frac{2e^2(1-\beta^2)}{m_0\, y^2\, 4\pi\epsilon_0} \qquad F_x = F_z = 0 \tag{6}$$

from which we see that (in the laboratory frame) the force between the charges vanishes at the velocity of light.

This latter case could have been solved directly, since the particles are at rest with respect to each other,

$$\frac{d^2y'}{dt'^2} = \frac{2e^2}{m_0\, y'^2\, 4\pi\epsilon_0} \tag{7}$$

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The result for the particle spacing ($y'$) after a time (t') may then be transformed into the laboratory frame,

$$t = \gamma t' \qquad y = y' \tag{8}$$

The reader may justifiably question why any space is given here to a discussion of relativistic electrodynamics; the reason is that the now-accepted formulation has been so recently decided, at least in the English language literature. (5) That all this is not so obvious as it appears may be inferred from a review of recent papers by authoritative sources on the radiation from a uniformly accelerated charge; these papers are about equally divided on the decision that such particles do or do not radiate. A particularly interesting, apparently unpublished, paper by D. Boulware discusses the question and the subtle reasons for the division of opinion.(6) The conclusion is, of course, interesting in the present problem because of radiation reaction on the particle bunch in an accelerator, Coleman's argument that the motion of a charge is unaffected by its own radiation of energy, seems incredible on the basis of conservation principles (7), even though there is no provision for it in the Lorentz force equation in the form

$$\frac{d\bar{p}}{dt} = e(\bar{E} + \bar{v}/c \times \bar{B}) \tag{9}$$

External Effects

Before considering the history of the cloud of electrons that constitutes a single pulse it may be worthwhile to consider some of the external disruptions that the beam may suffer in transiting the waveguide.

In medium energy, high current linacs it is an advisable practice to imbed the entire beam line in a solenoidal magnetic field, the intensity of which is determined by an attempt to provide Brillouin flow conditions; that is, the radial electric field of the charges and their 'centrifugal force' (when rotating in the field) is balanced by a centripetal force owing to the circumferential motion of the particles in the magnetic field. The circumferential motion of the beam particles is caused by the beam crossing the radial component of the solenoidal field at entry. The derivation of the condition is widely known, having been developed for microwave tubes and is given by

$$B_z = \frac{8.30}{b}\sqrt{\frac{I_o}{\sqrt{V_o}}} \tag{10}$$

where b is the beam radius, Io the current and Vo the beam voltage (all in MKS units with B in gauss). The relativistic derivation follows an identical line of reasoning and requires

$$B_z = \frac{0.369}{b}\sqrt{\frac{I}{\sqrt{\gamma^2-1}}} \tag{11}$$

where $\gamma$ is the normalized energy of the beam particles ($\gamma = 1 + eVo/m_oc^2$). In both of the above derivations the self magnetic field of the beam at its envelope is included. The two conditions are nearly the same.

However, it is also obvious that the 'corkscrew' rotation of the beam about its own axis will produce an induced magnetic field within the beam, which by Lenz's law will be opposed to the applied field thereby weakening the field appropriate to Brillouin flow.

It is also apparent on closer study that uniform space charge density is requisite to achieving perfect Brillouin flow, and this condition which initially depends upon the electron gun must be maintained in the presence of modulation or turbulence in the beam, otherwise the electrons will execute radial oscillations (i.e. the beam envelope will be scalloped), which is of course what happens.

The axial velocity of the beam is also modified by the flow condition since the circumferential motion represents an energy at the expense of the axial motion,

$$\frac{1}{2}m(r\dot{\phi})^2 + \frac{1}{2}m\dot{z}^2 = eV \tag{12}$$

from which,

$$\dot{z}^2 = 2\frac{eV}{m} - \left(\frac{eB}{m2}\right)^2 r \tag{13}$$

since $\dot{\phi}$ = eB/2m is for the Brillouin condition the rotational (or Larmor) frequency. The above observations are of interest in the injection region.

Additionally, if there is a transverse perturbation on an otherwise perfectly Brillouin focussed beam (such as may be caused by RF coupler asymmetry, the earth's field or induced modes in the waveguide) the beam trajectory will be helical (that is, the beam will 'corkscrew' down the beam aperture). This is the principle of the Cuccia coupler (8) and is known to the linac operator as the case where he cannot steer the beam 'rationally'; steering an off-set beam with what appears to be the appropriate steering coil results in an unexpected motion of the beam.

There are several other artifacts of the applied magnetic field worth mentioning: (1) If there is a radial gradient of the B-field (i.e. B changes appreciably in a distance of the order of the radius of gyration of the particles forming the beam) there will be an expansion of the beam. In the first approximation the electron spirals around the axis of the B-field in a manner (radius and frequency) determined by the local value of magnetic induction. With a transverse gradient of B, the next approximation to the electron motion is a sidewise drift of the beam guiding center of the particle orbits in a direction perpendicular to both B and ∇B. (2) Another type of field variation which causes a disturbance to the beam is curvature of the lines of force, such as will be caused by finite length solenoids (near the ends of the coil). Particle motion can be shown to be in a direction opposite to the sense of the curvature (and therefore alternately outward and inward in transiting a finite length coil). This latter effect is the same as the thick lens affect in long solenoids.

Experimental Analysis

For the purpose of the discussion which is the title of this paper it is intended to analyze some of the single pulse data reported from the ISIR linac, Osaka Univ. (8). Pulse length data was taken by means of a Čerenkov cell and streak camera, located 8 m (25.6 ft) beyond the end of the linac. At nominally 30 MeV a 16 nC pulse of 40 psec duration was recorded. Evidently the elapsed laboratory time for the bunch to travel the 8 m at 30 MeV ($\gamma$ = 59) is 27 ns, corresponding to a time interval of 0.45 ns in the moving frame of the bunch. This corresponds to the observation from the bunch that the distance to the Čerenkov cell is 0.14 m, so the time to reach it (or vice versa) is 0.45 ns. In this time the beam will not expand perceptibly in its own frame.

A closer scrutiny of the streak camera recording shows that about half the charge in the pulse exists in the FWHM time of 40 psec (19 deg phase interval at 1300 mcs.) and that the entire pulse lasts 172 psec (80 deg phase interval). The length of the dense central core of the bunch may be estimated from the transit time of Čerenkov cell as 1.2 cm axially. From beam spot size data this is also approximately the beam diameter. Thus, the central core of the pulse may be considered as a sphere or cylinder (of length equal to its diameter) as a first approximation. From the energy spectrum data the FWHM energy spread of the whole pulse was 0.035 with a maximum charge at 28.7 MeV implying that the bunch core phase was 17 deg off the wave crest, presumably ahead of the crest to take advantage of the sense of the field to cancel some space charge forces.

One can estimate the space charge field intensity at the edge of the central core,

$$\mathcal{E}_{sc} = \frac{q}{4\pi\epsilon_0 r^2} \tag{14}$$

which for 8 nC (the charge in the central core) and 1 cm diameter, to be 2.9 MV/m.

The plan to ride forward of the wave crest at a phase $\delta$ so as to cancel the space charge forces, given a bunch length $\varphi$ of charge $q$ and field strength E requires from eq (14) that

$$E\sin\left(\delta + \frac{\varphi}{2}\right) - \mathcal{E}_{sc} = E\sin\left(\delta - \frac{\varphi}{2}\right) + \mathcal{E}_{sc} \tag{15}$$

or

$$\cos\delta = \frac{\mathcal{E}_{sc}}{E\sin\frac{\varphi}{2}} \tag{16}$$

where no sacrifice in spectrum width is the condition imposed by eq (15). This condition generally also requires too great a sacrifice in achievable energy gain so that a compromise is established, although there is insufficient diagnostic information from the machine (while in operation) to know what that is.

From the foregoing it appears that the length and width of the bunch are essentially the same in the laboratory system although there is insufficient information to establish the shape of the bunch experimentally. To illustrate a simple example we assume a spherical packet of electrons in the moving frame (primed) of total charge q and assume that on leaving the accelerator t = 0, r = ro and dr/dt = 0. In the frame of the packet (with respect to which the sphere of charge is stationary) particles at the edge of the bunch experience a repulsive force and the sphere will expand in a manner described by the 'classical' equation of motion,

$$\frac{d^2r'}{dt'^2} = \frac{q}{4\pi\epsilon_0 r'^2} \tag{17}$$

Integrating twice,

$$\sqrt{\left(\frac{r'}{r_0'}\right)^2 - \frac{r'}{r_0'}} + \frac{1}{2}\ln\frac{1+\sqrt{1-\frac{r_0'}{r'}}}{1-\sqrt{1-\frac{r_0'}{r'}}} = \sqrt{\frac{2qe}{4\pi\epsilon_0 m_0 r_0'^3}}\; t' \tag{18}$$

To determine what the entire sphere looks like in the laboratory frame involves a somewhat complicated transform, but for the transverse and longitudinal cases the results are simple. In the transverse plane dimensions are not affected but time is contracted, so that t' = t/$\gamma$. In the axial direction, both the dimensions and time are contracted so that in the laboratory frame for the transverse diameter:

$$\sqrt{\left(\frac{r}{r_0}\right)^2 - \frac{r}{r_0}} + \frac{1}{2}\ln\frac{1+\sqrt{1-\frac{r_0}{r}}}{1-\sqrt{1-\frac{r_0}{r}}} = \sqrt{\frac{2qe}{4\pi\epsilon_0 m_0 c^2 r_0^3(\gamma^2-1)}}\; z \tag{19}$$

and for the axial diameter:

$$\left(\frac{r}{r_0}\right)^2 - \frac{r}{r_0} + \frac{1}{2}\ln\frac{1+\sqrt{1-\frac{r_0}{r}}}{1-\ \ 1-\frac{r_0}{r}} = \sqrt{\frac{2qe\,\gamma^3 r_0^{-3}}{4\pi\epsilon_0 m_0 c^2(\gamma^2-1)}}\; z \tag{20}$$

where the drift space has also been put in place of time,

$$t = \frac{z}{v} = \frac{\gamma z}{c\sqrt{\gamma^2-1}}$$

The conclusion is that the growth of the bunch is relatively greater in the axial direction than transversely over the same drift distance.

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