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A NEW MATHEMATICAL METHOD IN THE NON-LINEAR BEAM-BEAM INTERACTION.

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We here deal with a simplified version of the weak--strong beam-beam interaction that is present in storage rings: the two betatron modes x(t), y(t) of a test particle, in motion along its reference orbit and belonging to a low-density beam, are periodically coupled because of the periodic collisions with a much denser beam (we treat with bunched beams).

Let us consider, for definiteness, a case of two head-on collisions per revolution and with evident meaning of our symbols (x(t) stays for the horizontal mode, y(t) for the vertical one), we get:

$$\ddot{x} + \nu_{0x}^2\omega_0^2 x = -\xi x\cdot\phi(r)\cdot\sum_1^M{}_n\,\delta(t-n\,\tfrac{T}{2}) \tag{1}$$

$$\ddot{y} + \nu_{0y}^2\omega_0^2 y = -\xi y\cdot\phi(r)\cdot\sum_1^M{}_n\,\delta(t-n\,\tfrac{T}{2}) \tag{2}$$

where T is the period of revolution, equal to $2\pi/\omega_0$; $\phi(r)$ is equal to $(1-\exp\{-r^2/2\sigma^2\})/(r^2/2\sigma^2)$ if the charge distribution in the denser beam is gaussian and cylindrically symmetric (r.m.s.$\equiv\sigma=\sigma_x=\sigma_y$); ξ is the coupling parameter between two modes, is simply related with the number of particles N in the bunch [$\xi=(2r_eNC)/(\sigma\gamma^2)$ for e^+,e^- rings] and with the linear tune shift: $\xi = -2\pi\nu_0\Delta\nu$. M stays for the number of interactions ("kicks") and, being realistically of the order of 10^{10}, 10^{11}, can be assumed to tend to ∞. The case of a forced oscillator coupled with itself (one dimensional case) in which only the vertical mode y(t) feels the beam-beam force, makes sense physically as the simplest model that represents the effects of the collisions between a test particle and an unbunched beam. It requires a different function ϕ:

$$\phi(y) = \frac{2}{\sqrt{\pi}}\int_0^{y/\sqrt{2}\cdot\sigma} e^{-t^2}\,dt.$$

The integration of eqs.(1), (2). We search for solutions x(t), y(t) that are represented by the sequences $\{x_1(t), x_2(t),\ldots,x_k(t)\ldots\}$, $\{y_1(t),y_2(t)..y_k(t)..\}$ such that x(t), y(t) coincide with $x_n(t)$ and $y_n(t)$ respectively in the interval $(n-1)(T/2) \leqslant t \leqslant n(T/2)$ and are continuous. Furthermore, because of the delta function time-dependence of the interaction term, in the open interval $[(n-1)(T/2),\ n(T/2)]$ $x_n(t)$, $y_n(t)$ are free harmonic oscillator solutions:

$$x(t) = x_n(t) = A_n e^{i\nu_{0x}\omega_0 t} + B_n e^{-i\nu_{0x}\omega_0 t} \tag{3}$$

$$y(t) = y_n(t) = C_n e^{i\nu_{0y}\omega_0 t} + D_n e^{-i\nu_{0y}\omega_0 t} \tag{4}$$

Reality conditions for x(t), y(t) give $B_n=A_n^*$, $D_n=C_n^*$. By integrating eqs. (1), (2) in the interval $[n(T/2)-\varepsilon,\ n(T/2)+\varepsilon]$ with $\varepsilon\to 0^+$ and imposing the continuity of x(t), y(t) we finally obtain, after trivial manipulations:

$$A_{n+1}=A_n+i\,\frac{\xi}{2\nu_{0x}\omega_0}\,e^{-i\nu_{0x}\omega_0 n(T/2)}\,x_n(n\tfrac{T}{2})\cdot\phi[r_n(n\tfrac{T}{2})] \tag{5}$$

and similarly for C_n with y instead of x; $r_n=\sqrt{x_n^2+y_n^2}$ and A_1, C_1 being the initial conditions.

Interpolation and extrapolation with rational functions. The knowledge of the coordinates x(t), y(t) when t becomes of the order of 10^{10}, 10^{11} times the period T or, equivalently, the calculation of (A_n,C_n) for $n \sim 10^{10}, 10^{11}$ seems practically impossible to be achieved, step by step, on a computer. Instead we can reach easily the values of the sequence (A_n,C_n) when the integer n ranges between 1 and 10^6, 10^7. It then appears necessary to have a reliable extrapolation procedure at our disposal in order to use in a constructive manner the information that is present in the calculated part of the sequence and infer the values of (A_n,C_n) having n within the desired range (we may recall here that n can be regarded as a discrete time variable).

An algorythm that might respond to this purpose, within acceptable error limits, is the extrapolation--interpolation performed with rational functions; it is a simplified version of the so-called N-point Padé approximants [1]. To be more precise, once we are given the sequence $S_1,S_2,\ldots S_k\ldots$, we consider the rational function

$$\frac{P(x)}{Q(x)} = \frac{p_0+p_1x+p_2x^2+\ldots+p_\ell x^\ell}{q_0+q_1x+q_2x^2+\ldots+q_m x^m} \tag{7}$$

verifying the condition $[P(x_i)]/[Q(x_i)] = S_i$ (i=1,2..n) such that $m+\ell+1 \leqslant n$ and $x_i \equiv i$ or $1/i$ (more complicated relations between x_i and the index i are also possible in principle as long as $x_i\neq x_j$ for $i\neq j$). Attempts to find the interpolant are usually some variation of solving the problem in the cross-multiplied form:

$$P(x_i) = Q(x_i)S_i \tag{8}$$

which reduces to a homogeneous linear system of equations in the unknowns $p_0,p_1,\ p_\ell,\ q_0\ldots q_m$. The uniqueness of the interpolation (Cauchy) is not a priori certain and several degenerate configurations are possible [2].

If we reduce our analysis to the diagonal or paradiagonal ($\ell=m$; $\ell=m\pm1$ respectively) approximants, a very practical procedure is to use as rational interpolants the truncated functions obtained from the continued fraction S(x):

$$S(x) = \nu_1(x) \tag{9}$$

$$\nu_k(x) = \nu_k(x_k) + \frac{x-x_k}{\nu_{k+1}(x)} \tag{10}$$

$$S(x) = \nu_1(x_1) + \cfrac{x-x_1}{\nu_2(x_2)+\cfrac{x-x_2}{\nu_3(x_3)+\ldots}} \tag{11}$$

Recently [3] more elaborated and reliable rational interpolation algorythms have been introduced.

Algorythms for the limit $n\to\infty$. The interpolation--extrapolation procedure can become extremely complicated in practical applications and one might look for a simplified, global but quicker answer to the question of the stability of the amplitudes (A_n,C_n). This can come from the analysis of the limit of the moduli sequence $(|A_n|,|C_n|)$ for $n\to\infty$ (time $\to\infty$). The algorythms that calculate the limit are based again on the Padé approximants (of the first and second type): we mention here explicitly those three ρ,r,ε that we used in our computer experiments on the beam-beam problem.

Let us consider eq.(7) with $x_j\equiv j$ and $\ell=m$, the extrapolation limit $x_j\to\infty$, namely the ratio p_ℓ/q_ℓ, can be calculated by using the ρ-algorythm [4]:

$$\rho_{-1}^{(j)}=0,\ \rho_0^{(j)}=S_j,\ \rho_{k+1}^{(j)}=\rho_{k-1}^{(j+1)}+\frac{k}{\rho_k^{(j+1)}-\rho_k^{(j)}} \tag{12}$$

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the sequences with even lower index k provide usually the desired result, if it exists, whereas the odd k ones have purely an internal constructive role in the method. One could use the complex formulation $S_j = |A_j|+i|C_j|$ instead of evaluating separately (A_j) and (C_j) for $j\to\infty$. For the case $x_j\to 0$ ($x_j\equiv 1/j$) Stoer[4] has given an algorythm which calculates p_0/q_0 for a stair-case sequence of the type II approximants. It is called the r-algorythm and is defined by:

$$r_{-1}^{(j)}=0,\ r_0^{(j)}=S_j,\ r_k^{(j)}=r_{k-1}^{(j+1)} = \frac{r_{k-1}^{(j+1)}-r_{k-1}^{(j)}}{\frac{x_j}{x_{j+k}}\left[1-\frac{r_{k-1}^{(j+1)}-r_{k-1}^{(j)}}{r_{k-1}^{(j+1)}-r_{k-2}^{(j+1)}}\right]-1} \qquad (13)$$

The ε-algorythm[5], that usually accelerates a sequence to its limit, is based (instead of the first two) on the Padé approximants of the first type and is derived, once we are given a certain sequence $\{S_j\}$ by constructing the approximants [n,n+j] from the expansion $S(x) = S_0+(S_1-S_0)x+(S_2-S_1)x^2+\dots$ and taking their values at x=1. The ε sequences are calculated as follows:

$$\varepsilon_{-1}^{(j)}=0,\quad \varepsilon_0^{j}=S_j,\quad \varepsilon_{k+1}=\varepsilon_{k-1}+\frac{1}{\varepsilon_k^{j+1}-\varepsilon_k^{j-1}} \qquad (14)$$

The limit $j\to\infty$ is given by a certain sequence labelled by an even lower index whereas the odd lower index refers to intermediate steps of the calculation. The relation with the Padé approximants is

$$\varepsilon_{2n}^{j} = \{[n,n+j\ ]\}_{x=1}\ .$$

We may notice that this algorythm is normally very powerful and can be used in a multidimensional version for a vector sequence $\{\vec{S}_j\}$ by introducing the Samuelson inverse of an m-dimensional vector $\vec{x}\equiv(x_1,x_2,\dots x_m)$: $(\vec{x})^{-1}=(\vec{x}\cdot\vec{x}^*)^{-1}\cdot\vec{x}^*$ with $\vec{x}^*$ complex conjugate of $\vec{x}$.

Being based on the theory of power series the ε algorythm may produce a convergent sequence (its limit is the anti-limit) from $\{S_j\}$ if this diverges. The latter property compels us, for our purposes, to use the algorythm only as a checking tool; this is however very interesting because of its independence from the first two.

Preliminary numerical tests and remarks on the results. We have tested systematically the methods proposed above by comparing, first, their predictions in two extreme cases (called a and b) where either the input data are chosen in such a way that we expect stability at all times (and also for $t\to\infty$, $n\to\infty$), because of the weakness of the coupling parameter ξ in particular, or the input parameters are so evidently absurd on physical grounds, that the phase space orbit reaches its stochastic behaviour quite rapidly and one then gets instability for the betatron motion (see figures 1 and 2). For this particular comparison we limit our analysis to the one-dimensional case $\{r(t)\equiv x(t)\}$, where the stochastic regions in phase space are not linked with one another (no Arnold's diffusion), still considering the same function $\phi(x)$ present in eqs. (1) and (2). The parameters for the case a and b are respectively:

a) $x_0\equiv 2|A_1|=\mathrm{Re}(A_1+B_1)=0.1$ cm $\quad V_0=0 \quad \xi=0.2$

$\nu_{0x} = 3.14142136$

b) $x_0 \equiv 2|A_1|=\mathrm{Re}(A_1+B_1) = 0.5$ cm $\quad \nu_0=0 \quad \xi=7.0$

$\nu_{0x} = 3.14142136$

The limit is determined in the following manner: first we calculate from eq. (5) the sequence $|A_j|$ with j running from 1 to 10^6; then after this finite sequence is memorized, we extract from it various finite subsequences with different criteria and for each of them we calculate the limit $j\to\infty$ by the three algorythms ε, ρ, r. One of the adopted criteria lies in selecting a sequence of "maximi moduli", among a certain group of elements with a prescribed rule; then we look for its limit for $j\to\infty$. It appears obvious that we accept the result as a true limit only if the number obtained remains constant no matter what subsequence and algorythm are used.

The "maximi moduli" criteria for the cases a) and b) are the following: we pick up the absolute maximum among 200, 250, 500, 2000 values for the 10^6 points of $|A_j|$ in succession from $|A_1|$ to $|A_{10^6}|$ thus obtaining four finite subsequences of 5000, 4000, 2000, 500 terms respectively; these are now the input ingredients for our three algorythms ε, ρ, r. For the cases a) and b) the result is clear and allows a simple interpretation: the former gives us always, for all different criteria, the same result ($\lim_{n\to\infty} 2\cdot|A_n| = 0.106143$) and we can say that the limit exists and has a definite calculated value, the latter does not allow undoubtedly any limit (for instance different inputs give different results and some algorythms give absurd numbers). Therefore our procedure can discriminate, in the situation described above, very clearly between a stable and an unstable regime or, equivalently, between a regular orbit on a "torus" and a stochastic one in phase space.

For the case a) if we compare the "maximus modulus" limit with the one derived from other selections we may observe that the former is reached more quickly and is greater of a few per cent (between 2 and 4); furthermore it also guarantees us against eventual undesired large oscillations of the amplitudes $|A_j|$ for j large. Other sets of input parameters have been chosen in order to look at the limit: an analysis with ε and ρ only, based on 2000 terms extracted from the first 10^5 elements of the sequence $|A_j|$ and chosen for j running in succession from 23.001 to 24.999, from 48.001 to 50.000, from 73.001 to 75.000, and finally from 98.001 to 100.000, gives the information that the sequence seems to tend to a finite limit for parameters $x_0 \le 1.0$ and ξ between 0.2 and 2.0. This analysis is however incomplete, less clear and based on less input data (and two algorythms only).

Let us now come to the two-dimensional case $\{|A_j|,|C_j|\}$ j=1,2... . A detailed analysis has been performed on two physical situations characterized respectively by the parameters: $x_0=y_0=0.05 \quad \xi=0.1$ $\nu_{0x}=3.14142136 \quad \nu_{0y}=3.17283574$ and same x_0, y_0, ξ, ν_{0x} but $\nu_{0y}=2.14142136$. In both cases 10^6 values of $|A_j|$ $|C_j|$ are calculated and memorized from j=1 to j=10^6 with the eqs. (5). For the first case we pick up again subsequences of 1000, 2000 elements in various manners (either random or with fixed steps 1 every 1000 and 1 every 500 or blocks in succession of 1000 each) and in particular with the max.mod. criterion (max.mod. on 500, 1000, 2000 blocks). The three methods ε, ρ, r (ε and ρ used in the coupled two dimensional vectorial form) converge quite clearly, and rapidly in the max.mod.case, to the same value: we have $\lim_{j\to\infty}$ max.mod.$\{2|A_j|,\ 2|C_j|\} = \{0.05132, 0.05174\}$.

The second case (which has a resonance condition inbedded in $\nu_x-\nu_y=1$) is treated similarly and leads to an unambiguous limit (with ε, ρ, r):

$$\lim_{j\to\infty} \text{max.mod.}\ \{2|A_j|\ ,\ 2|C_j|\} = \{0.05328,\ 0.05238\}\ .$$

Other two dimensional limits have been computed on 100.000 values: the results are obviously less conclusive but in general seem to promise a regular behaviour, that is consistent with a limit, for $x_0=y_0 \lesssim 1$ cm, $\xi \lesssim 2.0$.

We end this note by emphasizing the fact that more detailed analysis based on actual physical parameters seems necessary in order to be conclusive on the capability of our approach but the discrimination pointed out before, between regular and stochastic regime, seems promising and encourages further investigations.

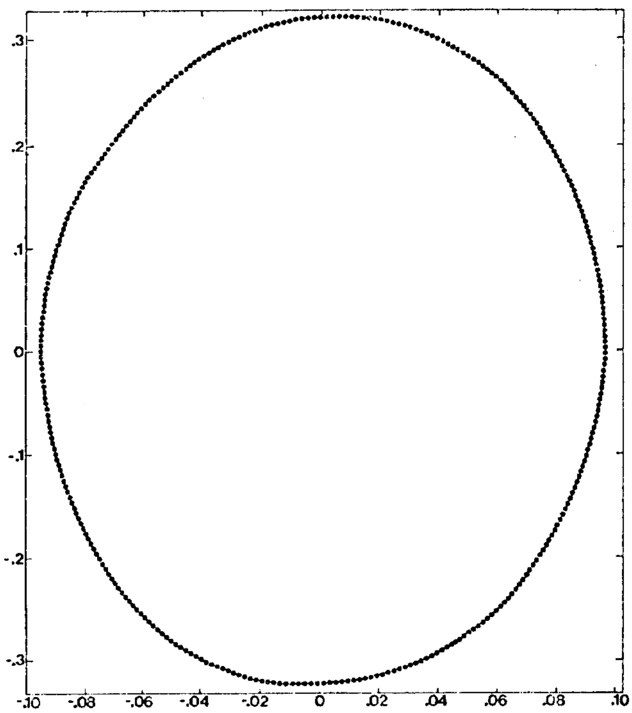

Fig.1 - Graphycal analysis in phase-space of the betatron oscillation coordinate x(t) for $x(0) = .1$ cm, $\xi = .2$

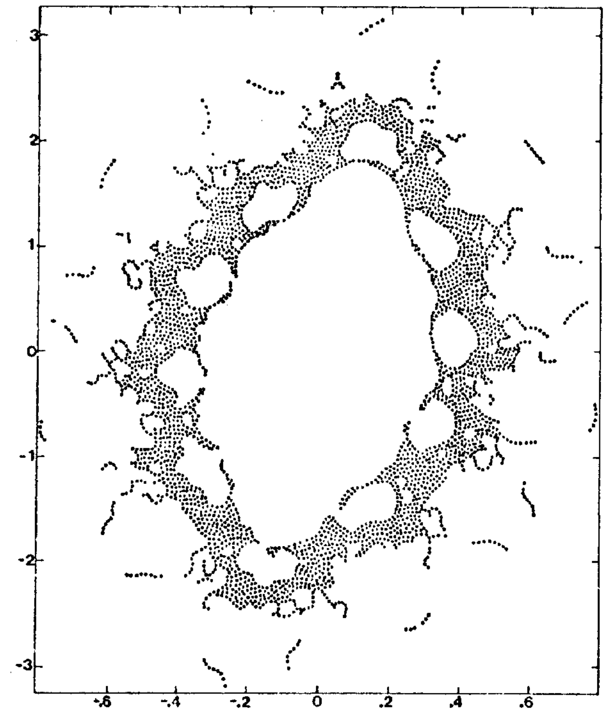

Fig.2 - Graphycal analysis in phase-space of the betatron oscillation coordinate x(t),for $x(0) = .5$ cm, $\xi = 7$.

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