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NON-LINEAR AND DISPERSIVE EFFECTS IN THE PROPAGATION AND GROWTH OF LONGITUDINAL WAVES ON A COASTING BEAM*

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Summary

The use of an intense beam of heavy ions to ignite a thermonuclear pellet at a distance from the final lens system requires a long thin beam with a very small longitudinal velocity spread. For perturbations long compared with the beam diameter, the longitudinal electric field due to the beam space charge can be approximated by the first derivative of the line charge density. This system of equations exhibits steepening of a finite amplitude wave. When other effects are included, such as the dispersion which occurs as the perturbation wavelength approaches the beam pipe diameter, a variation in density along the beam, or the exponential growth due to wall resistivity, the wave dynamics increase in complexity. Computer simulations are presented which illustrate some of the effects which can occur.

Introduction

For a long thin beam in a conducting pipe the longitudinal electric field due to space charge variations long compared with the pipe radius can be approximated by the first derivative[1] term in an expansion of derivatives of the beam line charge density. If the beam velocity spread is assumed small compared with the electrostatic wave velocity then the beam can be assumed cold and the dynamics described by,

$$\frac{\partial\lambda}{\partial t} + \frac{\partial\lambda v}{\partial x} = 0 \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{q^2}{m} g \frac{\partial\lambda}{\partial x} \tag{2}$$

where λ is the line charge density, $g = 1+2\ln(b/a)$, b is the pipe radius and a is the beam radius.

The waves obtained by linearizing this system of equations propagate at a phase velocity given by:

$$v_o^2 = \omega^2/k^2 = q^2 g\lambda_o/m = g\omega_p^2 a^2/4 \tag{3}$$

where ω_p is the plasma frequency in the beam. The linear eigenfunctions are characterized by:

$$\frac{v}{v_o} = \pm\frac{\delta\lambda}{\lambda_o} \tag{4}$$

In a space charge limited transport system, where the transverse focusing forces are balanced by the space charge repulsive forces in the transverse direction, the parameter $Q = \frac{\omega_p D}{\beta c}$, where D is the length of a focusing period, is approximately constant. For a 60° A-G transport system this constant is approximately 1.4. The wave velocity can then be written as

$$v_o/\beta c = \frac{1}{2}\sqrt{g}\,Qa/D.$$

The combination of continuity and momentum Eqs. (1) and (2) also describe a set of $\gamma = 2$ fluid equations. Finite perturbations in such a system are known to exhibit steepening[2]. This will characteristically occur when a disturbance of characteristic length L has traveled a distance of order L/α, where $\alpha = v_{max}/v_o \approx \delta\lambda_{max}/\lambda_o$ characterizes the amplitude of the perturbation.

As the wavelength of the perturbation decreases to the order of the pipe radius, the waves become dispersive. If the dispersion is approximated by the next term in the derivative expansion for the electric fields, so that $E \propto \frac{\partial\lambda}{\partial x} + C\frac{\partial^3\lambda}{\partial x^3}$ then the equation governing the system dynamics is the well studied Korteweg-de Vries equation, and a host of non-linear features such as the possible formation of solitons is expected.[3,4]

The general features of the Korteweg-de Vries equation which result from the presence of both steepening and dispersion can also be expected to occur when the dispersion is not simply represented by a third derivative term in the force law. A closer approximation[5] to the dispersive force law of a beam in a circular pipe is a force law whose k dependence is given by:

$$E \propto \frac{i\,k}{1 + (\frac{b}{2.4}k)^2} \tag{5}$$

The associated group velocity is then

$$v_g = v_o\,(1 + (\frac{b}{2.4}k^2)^{-3/2} \tag{6}$$

which tends to zero in the short wavelength limit.

When a longitudinal resistive force is included, the group velocity remains unchanged to lowest order in resistance, but exponential growth occurs at all k.

Propagation on a Finite Bunch

In the absence of non-linear and dispersive effects, and neglecting wall resistance, a localized perturbation on a beam will generally consist of a linear combination of forward and backward travelling eigenfunctions in the beam frame which separate and travel in opposite directions at v_o, thereafter retaining their shape. In the examples that follow, an initial perturbation in velocity is assumed with the functional form:

$$v = \frac{\alpha v_o}{1 + (\frac{x}{x_o})^2} \tag{7}$$

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Since the assumed initial conditions are not a stationary solution of the non-linear equations, a perturbation of this form will generally break up into a train of solitons as it propagates. In addition, because of the dispersion, larger k components of the initial disturbance will lag behind and a train of waves is observed even for a small amplitude initial perturbation. These deviations from the simple linear non-dispersive picture of a disturbance splitting into oppositely travelling eigenfunctions are likely to be most significant either when a disturbance has travelled a long distance or when it has a large amplitude.

The parameter range in a real accelerator often does not permit a perturbation to travel for a long distance before it encounters the end of the bunch. As the disturbance enters a region of decreasing density its propagation slows, wavelength decreases, and amplitude grows. The resulting behavior therefore becomes strongly dependent in the details of the behavior of large perturbations with gradient lengths on the order of the pipe diameter. It is in this range that the longitudinal one-dimensional model may not be adequate. Nevertheless, examination of the qualitative behavior can at very least indicate areas of possibly significant behavior for further study.

Figure 1 shows the phase space and density plots of a bunch with a parabolic current distribution chosen so that the equilibrium space charge forces can be balanced by a focusing force which increases linearly from the center of the bunch. The orbits of a large number of particles are followed in their self-consistent electric fields using a particle in cell code. In all runs a guassian numerical smoothing factor is applied in k-space so that the simulation "particles" have a gaussian shape[6] whose magnitude drops by 1/e in one half cell. The dispersion parameter is chosen so that the beam is 24b long and the initial perturbation chosen so that α = .018 and x_o = 1.8b. The top graph is formed by plotting 1000 simulation particles in x-v phase space. The bottom graph is a plot of the corresponding number density obtained by counting the number of particles in 200 bins across the system. As the wave propagates towards the ends of the bunch it slows and steepens. Figure 2 shows the same system after a time in which the initial perturbation would have propagated 11 pipe radii were the beam uniform, having the peak density. The behavior to this point is not radically different from that seen in the uniform beam. As the perturbation actually reaches the end however, a complicated interference pattern is set up between the forward and reflected waves which then slowly travels back toward the center of the bunch. Figure 3 shows the system after a time in which the initial disturbance would have travelled 32 pipe radii on a uniform beam. The interference pattern is strongly exhibited.

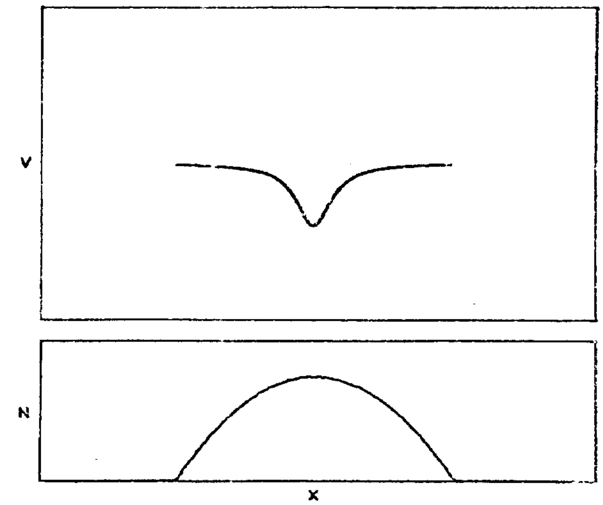

Fig. 1. Phase space and density of bunched beam showing initial perturbation.

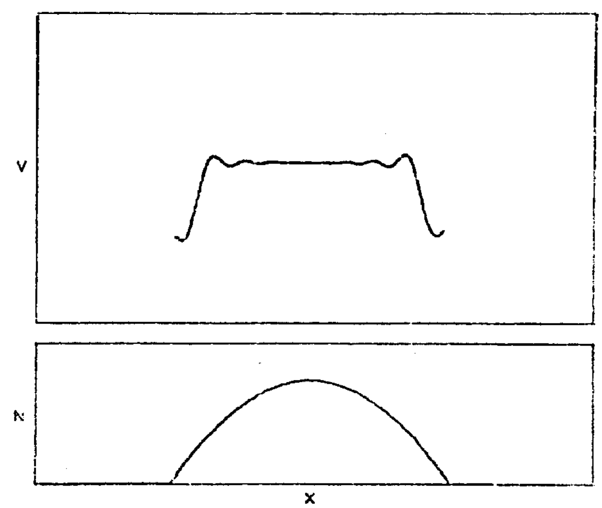

Fig. 2. Phase space and density of the initial perturbation shown in Fig. 1 after a time $11b/v_o$.

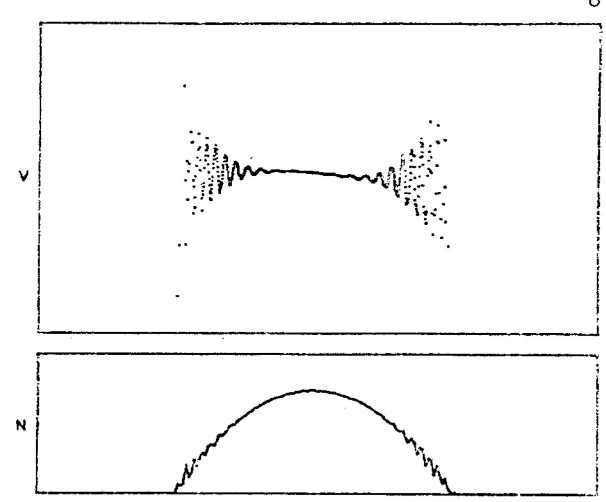

Fig. 3. Phase space and density after a time $32b/v_o$.

Simulations in which a larger initial disturbance is initiated show generally similar behavior except at the very end of the bunch. For sufficiently large amplitudes the density compression can, depending on the variation in the external focusing force outside the region of the initial current, ablate part of the beam.

Resistive Wall Effects

In the presence of the exponential growth caused by resistivity in the pipe wall, a small perturbation increasingly becomes subject to non-linearities as it grows until, at large enough amplitudes the beam can break apart. Even at small amplitudes, however, non-linearities can cause growth at greater than the linear growth rate.[7] Dispersion can also alter the low amplitude behavior.

Figure 4 shows an initial $\alpha = .018$ perturbation after it has travelled 16.7b. The initial perturbation has undergone 4.5 e-folds for its long wavelength components. This system represents for example a beam with a pipe radius of 21 cm with a resistance of 256 ohms per meter. In this case the perturbation would have travelled about 5 meters along the beam.

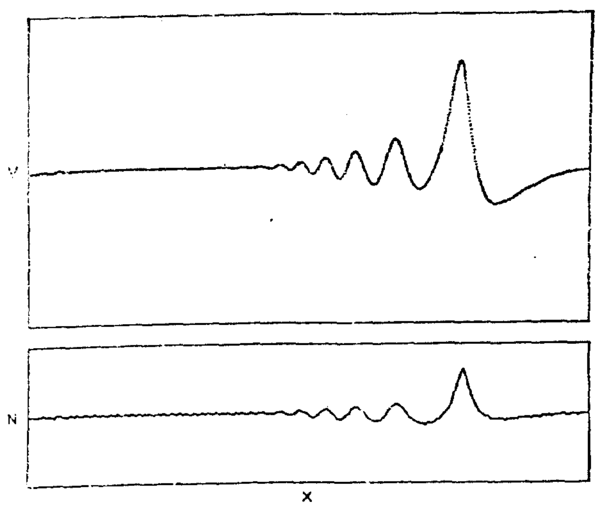

Fig. 4. Phase space and density of a uniform beam with resistive growth. The system is shown after a time $17b/v_o$, and 4.5 growth times.

Figure 5 is the same except that there is no dispersion. The behavior in this case is much closer to the conventional picture of a perturbation growing as it travels away from its initial position in the beam frame. The tendency for the dispersive system to "leave behind" the short wavelength components which travel at the slowest group velocities, can substantially reduce the mitigation of instability growth by convection of the perturbation to the rear end of the beam.

Conclusion

Some numerical calculations have been presented to illustrate the qualitative behavior of a long thin beam with very small longitudinal velocity spread when non-linear and dispersive effects are included. In the two examples presented the behavior is qualitatively different from what is predicted by simpler models. Furthermore, the assumption of a long cold beam is actually appropriate to the induction linacs and storage rings envisioned in heavy ion fusion driver systems.

A large amplitude perturbation, or a smaller disturbance either in a system with exponential growth or

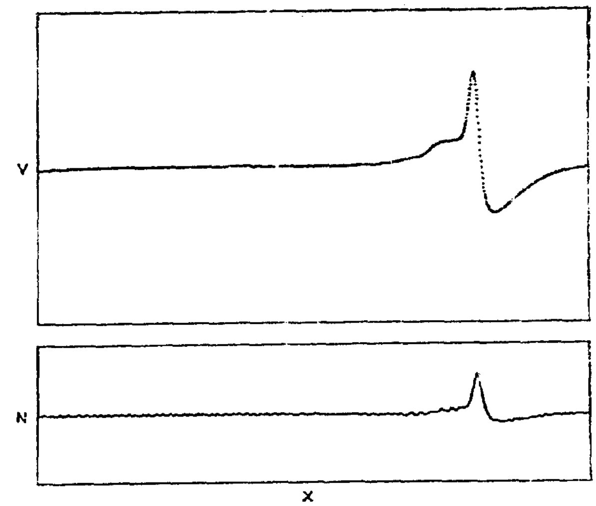

Fig. 5. Phase space and density under the same conditions as Fig. 4 but with no finite pipe diameter dispersion.

propagating down a density gradient, all show behavior strongly influenced by the beam dynamics on a scale length comparable to the pipe diameter. The quantitative behavior in this regime can be model dependent and an adequate model for the short wavelength regime should include multi-dimensional effects. Further study is therefore warranted. This does not however invalidate the conclusion that, in the parameter range studied, non-linear and dispersive effects can significantly influence longitudinal beam dynamics.

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