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BEAM-BEAM LIMIT AS A FUNCTION OF THE DAMPING TIME IN ELECTRON RINGS

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Summary

Observations on the interactions between beams colliding head-on have shown the importance of the non-linear resonances and the existence of a limiting beam strength. This limit is often associated with stochastic phenomena due to overlapping resonances. In this paper, a perturbation treatment[1] for isolated sum resonances is used. The forces are deduced from the space-charge potential of colliding beams[2]. Single resonance crossings are the consequence of tune modulations. Different crossing regimes, associated with trapping, statistical blow-up and mere beating are considered. For the parameters associated with the LEP project and the PETRA ring, an estimate of the speed of resonance crossing is made, in order to determine the crossing regime and the blow-up rate as functions of the linear beam-beam tune shift, ΔQ^{bb}, for each order N. The corresponding life time is compared to the damping time and the order of the safe resonances is given for different ΔQ^{bb}'s. Assuming a given value for the maximum order of resonances, which can be avoided in a certain region of the tune diagram, this allows a prediction of the beam-beam limit as a function of the damping time.

1. Theoretical Results

A particle of one beam sees the electric and magnetic fields due to the moving charges of the other beam. Since magnetic and electric forces are proportional, the perturbing Hamiltonian for beam-beam interactions may be written[1]:

$$H_1 = (1 + \beta^2) \frac{R^2 \quad \Phi}{\beta c \quad |B\rho|} , \qquad (1)$$

where $\beta = v/c$, R = machine radius, $|B\rho|$ = magnetic rigidity and Φ = electric potential. For a Gaussian beam, the space-charge potential is given by[2]:

$$\Phi(x,y) = \frac{I}{4\pi \varepsilon_0 \beta c} \sum_n \frac{(-1)^n}{2^n n!} \sum_{j_1+j_2=n} \binom{n}{j_2} \left(\frac{x}{\sigma_x}\right)^{2j_1} \left(\frac{z}{\sigma_z}\right)^{2j_2} J(\alpha,j_1,j_2), \qquad (2)$$

where:

$$J(\alpha,j_1,j_2) = \int_0^\infty \frac{dt}{(1+\alpha t)^{j_2+\frac{1}{2}} (1+t/\alpha)^{j_1+\frac{1}{2}}} ,$$

with I = beam current, σ_y = transverse r.m.s. beam size (y represents either x or z) and $\alpha = \sigma_x/\sigma_z$. The J integrals can be calculated by using recursion formulae[2] and the transverse beam size is related to the emittance E_y and the β_y-value at crossing point by $\sigma_y = \sqrt{E_y \beta_y^* / 4\pi}$.

Starting from eqs. (1) and (2), the usual perturbation treatment can then be done and isolated resonances, characterized by the relation $e = n_x Q_x + n_z Q_z - p = 0$ can be defined (Q_x and Q_z being the transverse tunes and $N = n_x + n_z$ the resonance order). The resonances are often described[1] by using the total bandwidth Δe and the octupole stabilizing coefficients, associated with a particle which oscillates within a transverse envelope of amplitude r_y ($r_y^2 = R E_y/2\pi$, if E_y is the emitttance of this oscillation) and phase ϕ_y. These parameters become for beam-beam forces:

$$\Delta e = \frac{(1+\beta^2) R \, r_x^{(n_x-2)} r_z^{(n_z-2)}}{\pi \beta c |B\rho| n_x! n_z! (2R)^{N/2}} \left(n_x^2 r_z^2 + n_z^2 r_x^2\right) \int_0^C \beta_x^{n_x/2} \beta_z^{n_z/2} \frac{\partial^N \Phi}{\partial x^{n_x} \partial z^{n_z}} \exp\left[i \left(n_x \mu_x + n_z \mu_z - e \frac{s}{R}\right)\right] ds , \qquad (3)$$

$$h_{20}^{(4)} \equiv h_{22000}^{(4)} = \frac{\alpha + 1}{4\alpha r_x^2} J(\alpha,2,0) \Delta Q_x^{bb} ,$$

$$h_{02}^{(4)} \equiv h_{00220}^{(4)} = \frac{\alpha + 1}{4 r_z^2} J(\alpha,0,2) \Delta Q_z^{bb} ,$$

where C = machine circumference.

Let us assume that the transverse tunes are modulated at the same frequency $Q_m f_{rev}$, with an amplitude ΔQ_y, i.e. $Q_y = Q_{y0} + \Delta Q_y \cos(Q_m \theta + \delta_y)$. This modulation induces periodic crossings of the betatron resonances and different regimes have to be considered, depending on the speed of the tune variations and on the perturbation strength. These regimes have to be studied from the equations of motion[1] in the variables (r_y, ϕ_y), which depend upon the phase $\psi_0 = n_x \phi_x + n_z \phi_z + e\theta$.

The transverse blow-up is governed by the change of ψ_0 with θ. In the neighbourhood of the point where ψ_0 is stationary ($\psi_0' = 0$), the change of ψ_0 is given by ψ_0'' and the phase remains constant when $\psi_0'' = 0$. In this case, the particle is trapped by the resonance and the explicit expression of ψ_0'' gives the trapping condition

$$Q_m \left| n_x \Delta Q_x + n_z \Delta Q_z \right| \leq 2 \left| \Delta e \, r_x^2 r_z^2 \frac{n_x^2 h_{20}^{(4)} + n_z^2 h_{02}^{(4)}}{n_x^2 r_z^2 + n_z^2 r_x^2} \right| . \qquad (4)$$

Equation (4) means that, for a given crossing speed, trapping only occurs for strong resonances and low-order resonances.

When trapping does not occur, the phase ψ_0 can be replaced by a series of phases $\psi_\ell = \psi_0 + \ell Q_m \theta$, where ℓ is an integer. Each term corresponds to a well-defined frequency provided that the derivatives ϕ_x' and ϕ_z' be constant. On the contrary, if the change in frequency is approximately equal to Q_m, there is a mixture between all terms, and the blow-up can be calculated on a statistical basis. The explicit form of ϕ_y' and of the amplitude change during one crossing give the following condition for obtaining a statistical blow-up:

$$Q_m^2 \leq 4\pi \, \Delta e \left| j_\ell \left(\frac{n_x \Delta Q_x + n_z \Delta Q_z}{Q_m}\right) r_x^2 r_z^2 \frac{n_x^2 h_{20}^{(4)} + n_z^2 h_{02}^{(4)}}{n_x^2 r_z^2 + n_z^2 r_x^2} \right| \qquad (5)$$

where j_ℓ is the Bessel function of order ℓ. The blow-up time constant τ^{bb} can be deduced from the amplitude growth due to a single resonance crossing, e.g. in the vertical plane,

$$\tau^{bb} \simeq \frac{4R(n_x\Delta Q_x + n_z\Delta Q_z)}{k^{n_x}\ \beta c}\left[\frac{n_z + kn_x}{\Delta e}\int_1^e \frac{dg_z}{\left(\frac{1}{k} - 1 + g_z^2\right)^{n_x/2} g_z^{(n_z-1)}}\right]^2 \qquad (6)$$

where $k = n_x r_z^2/n_z r_x^2$ and $g_z = r_{z,t}/r_z$, r_z being the initial vertical amplitude and $r_{z,t}$ the amplitude after a time t.

When condition (5) is not satisfied, each phase term ψ_ℓ corresponds to a well-defined frequency. The motion is then stable and small oscillations take place around the origin of the coordinates r_y and ϕ_y (or around some fixed point). In this case, the betatron resonance is replaced by a series of resonances $n_x Q_x + n_z Q_z + \ell Q_m - p = 0$, often called sidebands or satellites. The bandwidth of the sidebands are equal to $j_\ell\ \Delta e$, and there is no overlapping. In this stable regime, the amplitude beating is always limited.

2. Model Used for the Numerical Calculations

In electron storage rings, there are two mechanisms which change the tune of individual particles. On the one hand, the energy oscillations modulate the tunes at a frequency $Q_s f_{rev}$ with an amplitude $\Delta Q_y \simeq 1/2\ \xi Q_y\ \Delta p/p$, if Q_s is the synchrotron tune and ξ the chromaticity. On the other hand, the quantum excitation and the radiation damping change the betatron amplitudes and hence the tunes, since the beam-beam force is nonlinear. The frequency of this modulation can be approximated by the inverse of the damping time $1/\tau_d$ and the amplitude by $\Delta Q_y \simeq n_i\ \Delta Q^{bb}/2$ (n_i = number of interaction regions). Since the blow-up time constant (6) depends on the amplitude of the modulation and not on the frequency, the first mechanism, whose ΔQ_y is likely to be smaller, should predominate. Numerical calculations have, therefore, been performed for amplitudes ΔQ_y equal to and twice as large as those corresponding to synchrotron oscillations with $\xi = 0.1$.

The resonance parameters (3) depend on the oscillation amplitude r_y. The smaller the amplitude r_y, the more stable is the resonance. A reasonable choice for r_y seems to be the value which corresponds to a betatron amplitude of $2\sigma_y$. By this definition, when r_y remains stable, at least 86.5 % of the beam particles are stable. Calculations have been done for $2\sigma_y$ and the scaling law $\tau^{bb} \sim r_y^{2(2-N)}$ makes it possible to calculate the blow-up time constant at other amplitudes.

In a perfect machine, beam-beam forces excite the harmonics, which are multiples of n_i, and the resonances with even coefficients n_x and n_z (systematic resonances). Imperfections excite all resonances of any order (nonsystematic resonances). In our model, the imperfections (e.g. closed orbit distortions), which introduce errors in the relative centring of the two beams, have been considered. Since it seems difficult to control the vertical relative beam positions within ± 0.3 σ_z, an r.m.s. value of 0.15 σ_z has been taken for the vertical beam separation while the horizontal beam separation has been assumed to be smaller than 0.1 σ_x.

3. Numerical Applications

A computer program RESNOL, based on the formulae of Section 1, has been written for the calculation of the resonance parameters, the crossing regime limits and the time constants for statistical blow-up.

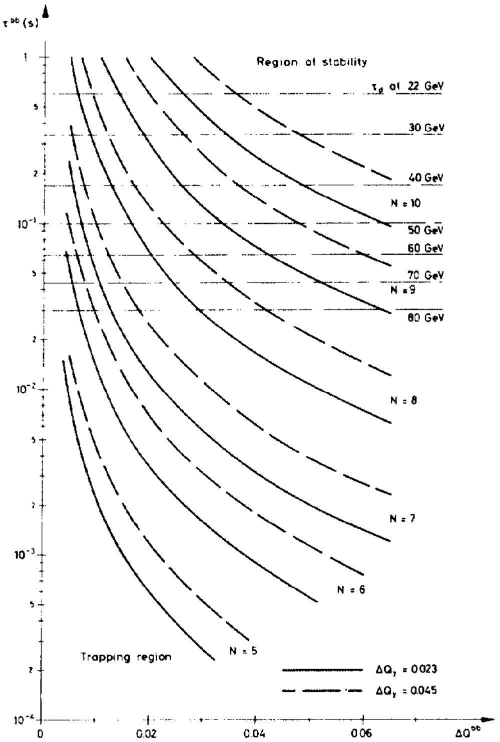

Fig. 1 Blow-up time constants vs. beam-beam tune shift for nonsystematic resonances in LEP, due to beam position errors.

The first application concerns the LEP project. The more recent design is for a machine with a radius of 4871.394 m, a phase advance per cell of 90°, tune values close to 97.75 and 101.75, a synchrotron tune close to 0.07 and transverse amplitudes $\sigma_x \simeq 0.3$ mm and $\sigma_z \simeq 0.02$ mm (β_x^*/β_z^* = 1.6/0.1 and 3.2/0.2). Figure 1 summarizes the results obtained. The blow-up time constant τ^{bb} is plotted as a function of the linear beam-beam tune shift. For resonances of order 6 or smaller (depending on ΔQ^{bb}), resonance trapping can take place, while stable oscillations appear for resonances of an order larger than 10. The curves of Fig. 1 (given for two different modulation amplitudes) are associated with the most critical nonsystematic resonance of each family of resonances of a given order N. The damping times τ_d at different LEP energies may be plotted on the same figure. Since the amplitudes are stable when $\tau^{bb} \simeq \tau_d$, Fig. 1 shows that, at 50 GeV, 10th order resonances can be damped for ΔQ^{bb} = 0.06, while 8th order resonances can be damped for ΔQ^{bb} = 0.02. In a more general way, it is possible to plot the value of ΔQ^{bb}, for which the blow-up is balanced by radiation damping,

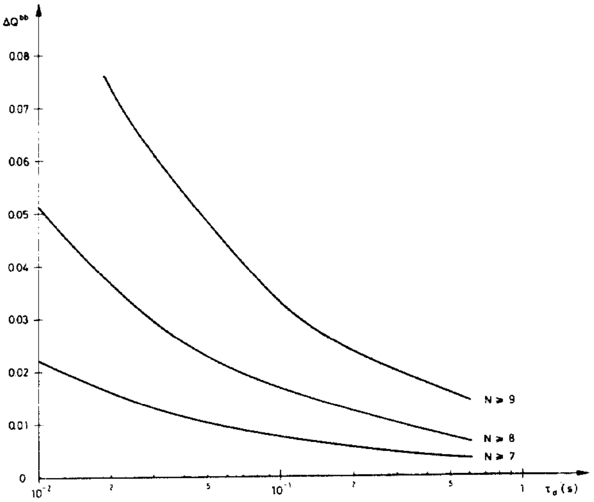

Fig. 2 Beam-beam limit vs. damping time in LEP for different thresholds in the resonance order N.

as a function of τ_d, assuming that only resonances of an order higher than a given value are crossed. This has been done in Fig. 2, which shows that increasing the threshold on N from 7 to 9 improves the tolerable ΔQ^{bb} by a factor of ~4.

The second application concerns the PETRA ring. Measurements have been made at an energy of 14 GeV for an optics with tune values close to 25.2 and 23.2, a synchrotron tune close to 0.06 and β_y^*-values, given by $\beta_x^*/\beta_z^* = 3.2/0.2$. Using these values and the theoretical parameters $\sigma_x \cong 0.64$ mm and $\sigma_z \cong 0.04$ mm, the above calculations were repeated and the results are summarized in Fig. 3. The curves, which have the same meaning as in Fig. 1, are given for a modulation amplitude of 0.005. The PETRA damping times (Fig. 3) indicate that, at 14 GeV, 10th order resonances are damped for $\Delta Q^{bb} = 0.06$, while 8th order resonances are damped for $\Delta Q^{bb} = 0.02$.

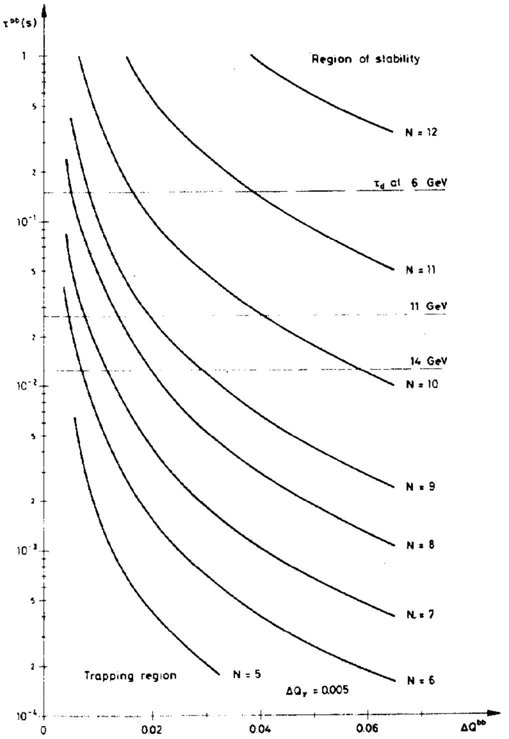

Fig. 3 Blow-up time constants for nonsystematic resonances in PETRA, due to beam position errors.

4. Conclusions

The curves in Figs. 1 to 3 show that linear beam-beam tune shifts between 0.01 and 0.02 might be reached at higher energy, if resonances of order 1 to 6 are avoided for particles with amplitudes equal or larger than $2\sigma_y$. This result agrees with the PETRA measurements[3], giving values between 0.01 and 0.024. If the working line of PETRA is in region 2 of Fig. 4, the particles with large amplitudes should only cross resonances of an order larger than 6. For LEP, working line regions near an integer (Fig. 4) could be envisaged. Region 2 is more suitable, since the larger the betatron amplitude, the higher is the order of the crossed resonances and since any blow-up shifts the line in a more favourable direction. Figure 4 shows, however, that linear tune shifts as large as 0.06 are difficult to achieve without crossing the strong 4th and 5th order resonances. Careful control of the beam positions, a decrease of the damping time and an increase of the octupole stabilizing coefficients should improve machine performance.

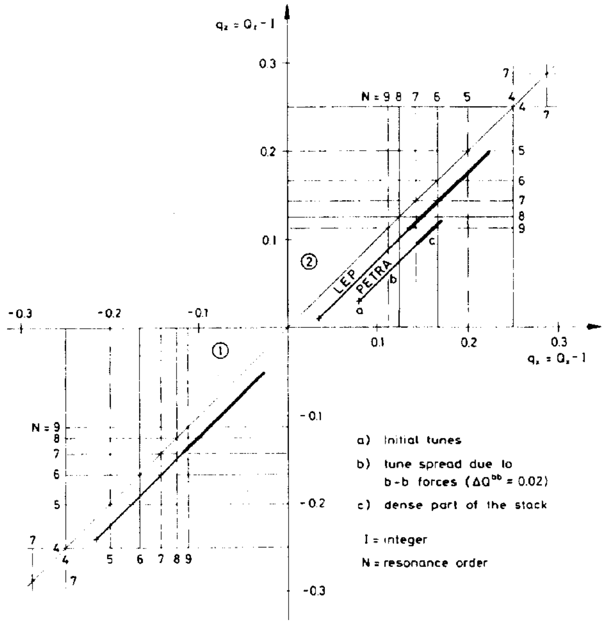

Fig. 4 Possible working lines in the tune diagram.

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