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"ARNOLD DIFFUSION" AND DIFFUSION ENHANCEMENT BY THE BEAM-BEAM INTERACTION

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Abstract

Results and analyses of computer simulations of the beam-beam interaction in the "Tevatron" $\bar{p}p$ collider are presented. Long-time simulations of this non-linear two-dimensional (2-D) interaction are undertaken in a search for beam blow-up due to "Arnold Diffusion". No large blow-up is seen in simulation of 20 minutes Tevatron time (60 million turns). In separate investigations it is found that the non-linear interaction can enhance diffusion from external sources, if a low order non-linear resonance be within the "tune spread". A model to describe this enhancement is presented.

Introduction

In the Fermilab Tevatron I project, bunches of protons and antiprotons (p's and $\bar{p}$'s) will collide in the 1000 GeV superconducting ring. As these bunches cross each other, particle motions will be perturbed by the non-linear "beam-beam" force from the opposite colliding bunch.

These $\bar{p}p$ collisions are different from those observed in earlier machines (e^+e^-, pp) and may have different limiting intensities. It has been suggested, for instance, that the "2-D" nature of $\bar{p}p$ collisions may cause beam loss due to "Arnold Diffusion". Also the beam-beam limits in e^+e^- and pp collisions are not fully understood and cannot be simply extrapolated to a $\bar{p}p$ collider.

To develop an understanding of beam-beam effects, we have undertaken simulations of beam-beam collisions. Results and analyses of these are presented in this paper.

Simulation Procedure: Equations of Motion

In the simulations reported in this paper, particle transport is calculated in three steps: a linear transport; a non-linear beam-beam kick; and a random diffusion kick. Particle motion through millions of these steps are calculated to simulate minutes of beam storage.

Particle motion from interaction region to interaction region is simulated by a linear 4x4 matrix, which is the usual Courant Snyder (C-S) matrix[1] characterized by the C-S parameters: ν_x, ν_y "tunes"; β_x, β_y, α_x, α_y. The two transverse motions (x and y) are not coupled and $\alpha_x = \alpha_y = 0$ is chosen at the interaction point.

The beam-beam interaction is simulated by adding a non-linear kick to the velocities x', y' of the form $x' \to x' + F_x(x,y)$. We use

$$F_x(x,y) = \frac{-4\pi\Delta\nu}{\beta_o} \; \frac{1 - e^{-\frac{(x^2+y^2)}{2\sigma^2}}}{\frac{x^2+y^2}{2\sigma^2}} \, x = -A\,\frac{\partial U(x,y)}{\partial x} \tag{1}$$

where $\Delta\nu$ is the "linear tune shift", β_o the C-S parameter, and this function corresponds to a round gaussian "strong" beam with rms radius σ. This beam-beam force is a "zero-length", "weak-strong" interaction. For the one-dimensional (1-D) cases with external "noise" this force is truncated by setting y = 0.

In these simulations only the fractional part of the tunes is significant; the integer part is set = 0. We note the distinctions among tune parameters: ν is the C-S tune of the transport between interactions, $\Delta\nu$ the "linear beam-beam tune shift", ν_o the tune of zero-amplitude particles ($\nu_o = \nu + \Delta\nu$).

Random diffusion is simulated by adding a random velocity kick on each turn to x', y':

$$x' \to x' + \theta_x R_x \tag{2}$$

where θ is a maximum amplitude and R is a random number between -1 and +1.

Our approximations simulate the special conditions of $\bar{p}p$ colliders: collisions of round beams, "weak-strong" collisions (more p's than $\bar{p}$'s), a short interaction region, no synchrotron radiation, but with some random diffusion dominated by beam-gas scattering or similar effects. For these colliders $\Delta\nu \lesssim 0.01$.

Long-time Simulation

Recent research has noted that 2-D motion in a non-linear force field can develop excursions to large amplitudes at large time scales, providing effectively unstable motion called "Arnold Diffusion"[2]. To determine whether such motion can occur within $\bar{p}p$ colliders we have undertaken long time scale simulations with high accuracy. In these cases random diffusion is not included and the high accuracy removes small errors so that we can isolate the "stochastic Arnold diffusion".

For these calculations CDC double precision is used to obtain 28 decimal place accuracy for a single turn calculation. Accuracy in a long time simulation was checked by following particle trajectories with different initial conditions forward in time for 60 million turns and then reversing the trajectories. On return, after 120 million turns (40 min. ring time), the initial and final positions agree to 14 decimal places, which indicates the scale of accumulated errors.

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The longest time scale simulation which we report here contains 60 million turns of calculations, which would correspond to 20 minutes storage ring in a $\bar{p}p$ collider with a 20 μs period.

For our cases we have followed the trajectories of 100 sample particles through 60 million turns and fitted the rms emittances ε_x and ε_y of this ensemble as a function of time to straight lines to obtain doubling times for that emittance.

Case I: $\nu_x = 0.245$, $\nu = 0.12$, $\Delta\nu = 0.01$. A 20 min. run shows an increase in $\varepsilon_R = \sqrt{\varepsilon_x^2 + \varepsilon_y^2}$ of 0.5% corresponding to a doubling time of 65 hours. The increase in ε_x of 1% is balanced by a decrease in ε_y of 0.25% (see Table I). Statistical analysis of our calculation procedures[3] indicates an expected error in this increase of ± 0.1% suggesting that these small changes may be significant.

Case II: $\nu_x = \nu_y = 0.245$, $\Delta\nu = 0.01$. A 20 min. run shows a decrease of ε_R of 0.03%, consistent with zero change. Both ε_x and ε_y show similar small decreases, consistent with zero change.

It has been shown[4] that if $\nu_x = \nu_y$, $\beta_x = \beta_y$, $\alpha_x = \alpha_y$ and the strong beam be round, then the quantity $p_\theta = x'y - y'x$ is constant, and the equations of motion can be reduced to 1-D and therefore should not show 2-D "Arnold Diffusion." This is consistent with our simulation (case II).

Although a similar invariant has been proven to exist[4] if $\beta_x \neq \beta_y$, we have not found any invariant with $\nu_x \neq \nu_y$. An increase in ε by "Arnold Diffusion" in case I is still possible and may explain the small charge we see.

In summary, our long-time simulations show that beam-beam collisions with $\bar{p}p$ parameters ($\Delta\nu \simeq 0.01$, round beams, ~1 day storage) should not show large blow-up due to "Arnold Diffusion" (see ref. 3 for details).

Diffusion Enhancement in 1-D Simulation

The random diffusion of eq. (2) causes an increase in rms emittance ε, defined by

$$\varepsilon = 6\sqrt{<(x - x_o)^2> < (x' - x'_o)^2>}$$

where x_o, x'_o are average values (beam centroid). In the absence of a beam-beam force a random kick causes a linear increase of ε with time t given by

$$\varepsilon = \varepsilon_o + D_o\, t = \varepsilon_o + \beta\, \theta^2\, n_T \qquad (3)$$

where n_T is the number of turns and D_o is a diffusion constant.

Our numerical simulations show that this increase can be substantially modified by the beam-beam force. The increase remains linear to a first approximation and we rewrite (3) as

$$\varepsilon = \varepsilon_o + \alpha_E\, D_o\, t$$

where α_E is a "diffusion enhancement factor". We have calculated α_E in 1-D simulations by following ~100 particle trajectories[5,6] for ~200,000 turns for various values of ν_o, $\Delta\nu$ and D_o and some of these results are displayed in Table 2. In these cases we have deliberately chosen $\Delta\nu$ and D_o larger than in physical $\bar{p}p$ colliders in order to develop observable effects within short computer times.

From these and other cases we derive some features of this enhancement:

1. α_E can be substantially greater than 1 if a major low order resonance (1/4, 1/6, 1/8 ...) be within the tune spread ν_o to $\nu_o - \Delta\nu$, with greatest enhancement by lowest order resonances.

2. α_E is independent of D_o.

3. The development of the rms increase in beam size depends on the location of the resonance within the tune spread. If the resonant tune be near ν, the enhancement is due to a few particles kicked to large amplitudes. If not, the enhancement is distributed more uniformly.

A Model to Describe 1-D Diffusion Enhancement

In Figure 1 we show sample particle trajectories for a particular case ($\nu_o = 0.20$, $\Delta\nu = 0.04$). Without a beam-beam interaction these trajectories are elliptical with the quantity $I = \sqrt{x^2/\beta + \beta(x')^2}$ invariant. The non-linear force modifies these trajectories so that those with amplitudes $I_o - \Delta$ and $I_o + \Delta$ are adjacent, where I_o is the center and 2Δ is the width of the resonant region with resonant tune $\nu_R = k/m$.

In the approximation that the particle motion is dominated by the lowest order resonant harmonics I_o and Δ can be expressed in terms of the Fourier harmonics of the beam-beam potential $U(x)$.

$$U_m(I) = \frac{1}{2\pi}\int_0^{2\pi} U(\sqrt{2I\beta_o}\cos\phi)\cos(m\phi)\, d\phi$$

where I is defined above and ϕ is defined by $x = 2I\,\beta_o\cos\phi$. I_o is the solution of:

$$\nu - \nu_R + \frac{A}{2\pi}\quad\frac{\partial U_o(I)}{\partial I} = 0$$

and

$$\Delta \simeq \sqrt{\frac{8\, U_m(I)_o}{U''_o\,(I_o)}} \qquad \text{where } U''_o\,(I) = \frac{\partial^2 U_o}{\partial I^2}$$

These relations are derived in ref. 5 which also includes numerical calculations of U_m for our gaussian potential.

Our model for diffusion enhancement is as follows[5]: Random diffusion increases particle amplitudes I in a Brownian manner. The resonance places the amplitudes $I_o - \Delta$ and $I_o + \Delta$ adjacent so that Brownian motion increases mean particle amplitudes by $\sim 2\Delta$ when particles reach the threshold $I_T \simeq I_o - \Delta$. The diffusion is therefore enhanced by a term proportional to the resonance width multiplied by the rate at which particles reach the threshold I_T. For gaussian beams that rate is

$$\frac{\dot{N}}{N} = 6 \; \frac{D_o \, I_T}{\varepsilon_o^2} \; \exp\,(-\,6\; I_T/\varepsilon_o)$$

Our diffusion constant is:

$$D \simeq D_o \left\{ 1 + 72 \; \frac{\Delta . I_T}{\varepsilon_o^2} \; \exp\,(-\,6\; I_T/\varepsilon_o) \right\}$$

In Table 3 we present comparisons of α_E calculated with the above model and as generated in our simulations. Both qualitative and quantitative agreement is obtained (see ref. 5 for details).

Conclusions

Our long-time simulations indicate that there is no large beam blow-up due to "Arnold Diffusion" at $\bar{p}p$ collider parameters. Future simulations will be undertaken to set further limits on this process.

Our 1-D simulations indicate that the coupling of random diffusion from outside sources to the non-linear force is important in limiting beam stability. Extension of our analysis to the $\bar{p}p$ 2-D geometry and inclusion of other non-linear and noise effects such as "tune modulation" and "power supply ripple" will make these limits more precise.

Our analyses to date indicate that $\bar{p}p$ collisions with tune shifts of $\lesssim$ 0.01 should be possible without loss of luminosity, even with storage times of many hours.

References

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Table I - Change of Emittance Values in Long Time Simulation

Case I - $\nu_x = 0.245$, $\nu_y = 0.12$, $\Delta\nu = 0.01$

T	ε_x	% Change per Hour	ε_y	% Change per Hour	ε_R	% Change per Hour
0 min.	.0227510	-----	0.0191872	-----	0.0298409	-----
8	.0227892	+6.9%	0.0192630	-6.0%	0.0298550	+1.3%
12	.0228320	+2.6%	0.0192638	-1.6%	0.0298883	+1.9%
16	.0228312	+2.0%	0.0192581	-1.2%	0.0298878	+0.7%
20	.0228630	+3.2%	0.0192567	-0.8%	0.0299076	+1.5%

Case II - $\nu_x = \nu_y \simeq 0.245$, $\Delta\nu = .01$

	ε_x		ε_y		ε_R	
0 min.	0.0237166	-----	0.0236805	-----	0.0336119	-----
8	0.0237051	-.02%	0.0237027	-.18%	0.0336262	+.08%
12	0.0237030	-.23%	0.0237047	+.12%	0.0336253	-.06%
16	0.0237014	-.25%	0.0237024	-.07%	0.0336232	-.14%
20	0.0237020	-.11%	0.0237022	-.01%	0.0336229	-.08%

Table II - Diffusion Enhancement in Simulations with 100 Particles, 200,000 Turns

ν	$\Delta\nu$	ν_R	D_o (Eq. 3)	α_E (Simulation)
.20	0.	--	.008	0.96
.17	0.03	--	.008	0.85
.16	0.04	1/6	.008	2.18
.16	0.04	1/6	.032	2.56
.10	0.10	1/6,1/8	.008	2.23
.26	0.04	--	.008	1.26
.24	0.06	1/4	.008	7.44

Table III - Comparison of Calculated Diffusion Enhancement with Simulation

ν	$\Delta\nu$	ν_R	Δ	$\frac{\dot{N}}{N} I_T$	α_E (Model)	α_E (Simulation)
.16	.04	1/6	.026	.043	2.7	2.3
.24	.06	1/4	.046	.12	9.1	7.5
.22	.08	1/4	.013	.14	3.7	4.0
.10	.10	{1/6, 1/8	.003, .006	.14, .003}	1.8	2.0

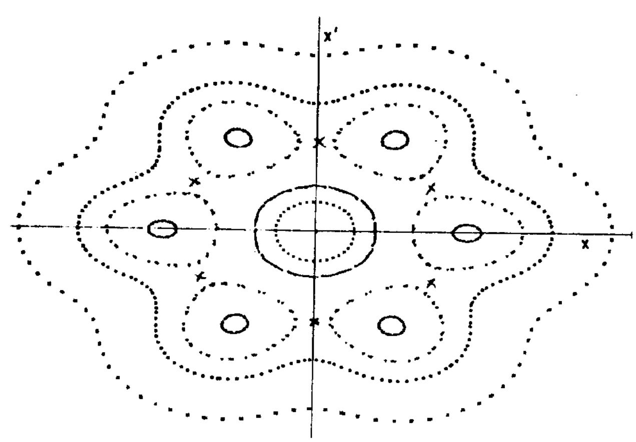

Fig. 1 Phase Space Trajectories in Presence of a Non-Linear Resonance (6-th order)