IEEE Transactions on Nuclear Science, Vol. NS-28, No. 3, June 1981

ORBIT DYNAMICS IN ISOCHRONOUS CYCLOTRONS

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Introduction

Acceleration of heavy ions in an isochronous cyclotron can lead to a ν_r-value near unity for many turns. In ORIC, several departures from 3-fold symmetry unfortunately exist, so that a complex resonant behavior may lead to beam loss. To study these questions, a program has been initiated with the aim of devising and using very efficient algorithms for calculating the motion of ions during acceleration. The eikonal function was chosen as the basis for these studies, with methods and some results here described.

Definition and Computation of the Eikonal

Basically, the eikonal is the generating function for the contact transformation which transforms the initial coordinates and momenta of a phase trajectory into the final coordinates and momenta. In particular, we assume

$$S(q_1 \ldots q_n,\ Q_1 \ldots Q_n) \tag{1}$$

as the generating function, with

$$p_i = \frac{\partial S}{\partial q_i}, \qquad P_i = -\frac{\partial S}{\partial Q_i}. \tag{2}$$

Lower case variables will be final values and upper case will be initial values. As normally used in optics, the independent variable is not time, but some variable measured along a central ray. Frequently, this variable would be arc length along the central ray, which in a centered optical system would be the optical axis.

In such a problem q_1 and q_2 would be transverse displacements, perpendicular to the central ray, and p_1 and p_2 would be the corresponding angles of inclination to the central ray. In the accelerator problem with medium plane, we naturally use q_1 as radial displacement from the equilibrium orbit and q_2 as the axial displacement from the medium plane. For reasons of computational simplicity, the independent variable will always be the polar angle θ, measured with respect to a pole well centered in the magnetic field.

Although the concepts here used are fully generalizable to the case of r- and z-motion, we have thus far restricted our study to radial motion only, since the behavior for $\nu_r \simeq 1$ should be dominated by the radial resonances $\nu_r = 1/1$, $2/2$, and $3/3$, the latter being an essential resonance in the (approximately) 3-sector ORIC.

As used in optics, the eikonal is particularly well suited to the problem of describing small aberrations from the central ray. For this purpose, we separate S into orders

$$S(q,Q) = S^{(2)}(q,Q) + S^{(3)}(q,Q) + S^{(4)}(q,Q) + \ldots$$

where (3)

$$S^{(2)}(q,Q) = S_1 q^2 + S_2 qQ + S_3 Q^2$$

$$S^{(3)}(q,Q) = S_4 q^3 + S_5 q^2 Q + S_6 qQ^2 + S_7 Q^3$$

$$S^{(4)}(q,Q) = S_8 q^4 + S_9 q^3 Q + S_{10} q^2 Q^2 + S_{11} qQ^3 + S_{12} Q^4$$

etc.

The function $S^{(2)}$ describes exact Gaussian imaging, $S^{(3)}$ describes the lowest order aberrations (Vanishes identically in a centered optical system), $S^{(4)}$ describes the next order of aberration (Seidel aberrations in a centered system).

These successive functions can clearly be calculated, in principle, by tracing a sufficient number of rays, but a serious practical problem arises from the necessity of taking the limit $q,Q \to 0$. Fortunately, these orders can be calculated simply and directly by use of the fundamental property

$$S(q,Q) = \int_0^{\theta_f} d\theta \ \ L(x(\theta), \dot{x}(\theta)) \tag{5}$$

where L is the Lagrange function of x and its derivative $\dot{x}$ with respect to the independent variable θ. Note that

$$x(0) = Q, \qquad x(\theta_f) = q \tag{6}$$

and that x in (5) is assumed to describe the actual trajectory which joins the assumed end values. In consequence, (5) might appear to have little practical utility. Actually L is normally easily separated into terms of definite order. Thus

$$L = L^{(2)}(x,\dot{x}) + L^{(3)}(x,\dot{x}) + L^{(4)}(x,\dot{x}) + \ldots \tag{7}$$

where

$$L^{(2)}(x,\dot{x}) = L_1 x^2 + L_2 x\dot{x} + L_3 \dot{x}^2$$

$$L^{(3)}(x,\dot{x}) = L_4 x^3 + L_5 x^2\dot{x} + L_6 x\dot{x}^2 + L_7 \dot{x}^3 \tag{8}$$

$$L^{(4)}(x,\dot{x}) = L_8 x^4 + L_9 x^3\dot{x} + L_{10} x^2\dot{x}^2 + L_{11} x\dot{x}^3 + L_{12}\dot{x}^4$$

etc.

It will be immediately recognized that the coefficients $L_1(\theta)$, $L_2(\theta)$, ... can be easily written down in terms of the median plane magnetic field, its successive radial derivatives, and $R(\theta)$ and $\dot{R}(\theta)$, the equilibrium orbit radius as a function of θ and its θ-derivative.

*Research sponsored by the Division of Basic Energy Sciences, U.S. Department of Energy under contract W-7405-eng-26 with the Union Carbide Corporation.

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The expression (5) can now be split into its order orders by use of the following procedure. We first define successive orders for the ray coordinate. Thus

$$x = x^{(0)} + x^{(1)} + x^{(2)} + \ldots . \qquad (9)$$

where

$$x^{(0)} = q\, u_1(\theta) + Q\, u_2(\theta)$$

$$x^{(1)} = q^2 v_1(\theta) + qQv_2(\theta) + Q^2 v_3(\theta) \qquad (10)$$

etc.

and

$$u_1(0) = u_2(\theta_f) = v_1(0) = v_2(0) = v_3(0) = v_1(\theta_f) = v_2(\theta_f) = v_3(\theta_f) = 0$$

$$u_1(\theta_f) = u_2(0) = 1.$$

The functions u_1 and u_2 satisfy the linearized equations of motion, derived from $L^{(2)}$ alone. The functions v_1, v_2, v_3 also satisfy the linearized equations, but with quadratic forcing terms derived by inserting $x^{(0)}$ in $L^{(3)}$.

By some manipulation involving integration by parts, and use of the variational connection between L and the equations of motion, we finally obtain explicit relations for $S^{(2)}$, $S^{(3)}$, and $S^{(4)}$

$$S^{(2)} = \int_o^{\theta_f} d\theta\, L^{(2)}(x^{(0)}, \dot{x}^{(0)})$$

$$S^{(3)} = \int_o^{\theta_f} d\theta\, L^{(3)}(x^{(0)}, \dot{x}^{(0)}) \qquad (11)$$

$$S^{(4)} = \int_o^{\theta_f} d\theta\, L^{(4)}(x^{(0)}, \dot{x}^{(0)}) - \int_o^{\theta_f} d\theta\, L^{(2)}(x^{(1)}, \dot{x}^{(1)}).$$

Relations of increasing complexity can be obtained for the higher $S^{(n)}$ but, for reasons to be discussed, we confine our attention to the three above.

The computational procedure to be used is the following. We choose an energy value and find the equilibrium orbit by a conventional search. The various θ-dependent coefficients L_n can then be calculated and stored. Another pass finds u_1 and u_2, and a slightly more complicated pass finds v_1, v_2, and v_3. Another pass evaluates $S_1 \ldots S_{12}$ and the computation is complete for that energy.

Computational Details

The measured median plane (available for ORIC on a polar mesh) is interpolated by a quintic spline routine and stored in a disk file with 81 r-values and 121 θ-values. This is a basic input file for the program SCAL, which implements the procedure we have described. SCAL uses the spline again to find the first three radial derivatives of the field, and then proceeds with the calculation outlined above. Integration is by the Runge-Kutta method, with 60 steps per revolution. SCAL can take θ_f as 2π, $2\pi/2$, or $2\pi/6$, but in any event, the coefficients are stored so that the complete 0-2π transformation can be constructed.

Uses of the Eikonal

The original intention was to use the 1-turn Eikonal to find an invariant function such that

$$H(q,p) = H(Q,P) \qquad (12)$$

for all values of Q and P, with the assumption that q and p are related to Q and P by the contact transformation defined by S. From the analytic behavior of H, it should be possible to obtain reliable information on stability limits and related matters. In the conventional treatment, severe simplification of L, with use of the phase-amplitude method, leads to an approximately invariant function, and it is a reasonable hope that the present treatment can yield an H which can accurately describe a real system.

For this purpose, a program HCAL was constructed to use as input $S_1 \ldots S_{12}$ and yield $H_1 \ldots H_{12}$, where

$$H = H^{(2)} + H^{(3)} + H^{(4)} + \ldots .$$

$$H^{(2)} = H_1 q^2 + H_2 qp + H_3 p^2$$

$$H^{(3)} = H_4 q^3 + H_5 q^2 p + H_6 q p^2 + H_7 p^3 \qquad (13)$$

$$H^{(4)} = H_8 q^4 + H_9 q^3 p + H_{10} q^2 p^2 + H_{11} q p^3 + H_{12} p^4$$

It turns out that this essentially algebraic task can be accomplished quite economically, but with some questions as to its full significance. Finding $H^{(2)}$ is a standard problem, and $H^{(3)}$ is then unambiguously determined by the requirement of invariance, plus probably a hidden requirement as to the analyticity of H(q,p) for small values of its arguments. $H^{(4)}$ is necessarily incompletely determined by the invariance argument, since an arbitrary amount of $(H^{(2)})^2$ can be added to it. It will be recognized that this arbitrary addition alters the quartic detuning term which always governs the behavior of the phase plot outside the simple cubic region. It is thus necessary to adjust the proportion of $(H^{(2)})^2$ to cause the mean circulation rate about the origin of the phase plot to be correct. Experience has shown that the stability limit as determined by $H^{(2)}$ and $H^{(3)}$ is quite adequate for a 3-sector machine, so that the above ambiguity becomes of possible practical importance only for a 4-sector machine.

We have made provision for dividing the full revolution into two half transformations, in order that acceleration by two gaps (0° and 180°) can be treated. When a program was written to exploit this approach, the results were extremely disappointing. Although the phase plots obtained from H were apparently accurate, use of the single S from 0 to 2π, or the two half-transformations yields phase plots which are only accurate at uselessly small oscillation amplitudes. Extensive testing strongly suggests that this is a matter of principle, which is not yet resolved.

To make further progress, it was noted that the transformation described by $S^{(1)} + S^{(2)} + S^{(3)}$ does not really offer much complexity of behavior. By splitting the range 0-2π up into a larger number of sub-ranges (called sectors in the following, although no true repetition is implied), it is clear that we quickly approach the solution of the full equations of motion. In order to fit the symmetry of the accelerating electrodes, it seemed reasonable to try six sectors, and much of the work has used this. Very good phase plots can now be generated by repetitive transformation, the process being still very

efficient. We show in Fig. 1 a typical phase plot for uranium ions at 2 MeV/amu in a field of the proposed ORIC superconducting upgrade. This curve was obtained by plotting contours of constant H.

In Fig. 2 is shown a parallel calculation by repetitive transformation. It is clear that the stability limits are similar, although some differences in definition between the two plots makes precise equivalence more difficult to establish.

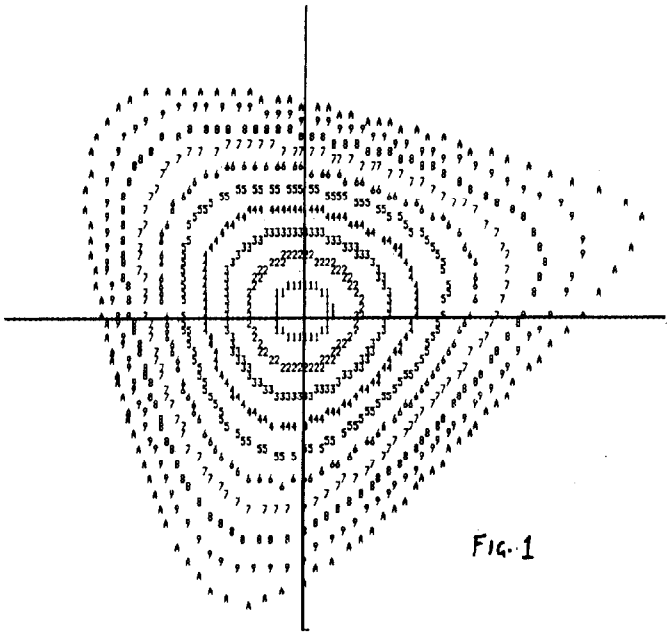

Fig. 1

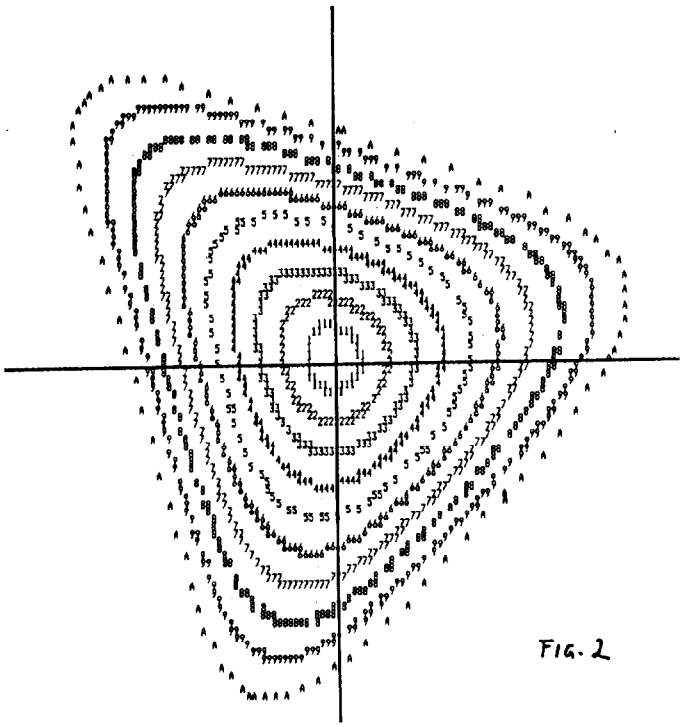

Fig. 2