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LONGITUDINAL INSTABILITIES IN CIRCULAR ACCELERATOR AND STORAGE RINGS*

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Introduction

Coherent instabilities of particle beams in circular accelerators and storage rings have been the subject of many theoretical and experimental works, starting from the early 1960's. I will not try to review all work done in nearly two decades and will limit myself to give references only to the most recent works or the works more strictly related to the particular way I describe these phenomena. An excellent review of the early work can be found in reference 1 and for some aspects more related to storage rings in reference 2.

It is usual to classify the coherent instabilities as either longitudinal or transverse according to whether they influence the synchrotron or betatron oscillations. The same qualitative description applies to both cases and for simplicity in this paper I will only consider longitudinal instabilities.

The coherent instabilities we want to discuss are produced by the electromagnetic interaction of the charged particle beam with the walls of the vacuum chamber in which the beam is moving. The wall geometry is complicated by the presence in the accelerator of the diagnostic equipment, radio-frequency cavities and other equipment necessary to operate the system. The field produced by the beam and modified by the walls causes a force, proportional to the beam current, acting on the beam itself and that can lead to what we call a coherent instability.

Since the wall geometry can be very different for the various accelerators it is very useful to follow the idea of Vaccaro and Sessler[3] of characterizing the vacuum chamber and all the associated devices, which we will call the "beam environment", by an impedance function $Z(\omega)$, relating the beam current and the electromagnetic field perturbing the beam. In this way we can study the beam dynamics in general terms and then apply the results to a particular accelerator by specifying the impedance.

Although one can write the general equations describing the beam dynamics and the stability properties, it is not possible to obtain a general solution of these equations for a realistic beam charge distribution and wall impedance. It is however possible to obtain solutions valid in certain regions of the parameter space of this problem.

Let us introduce a time scale using the characteristic times of the beam and its environment: the bunch duration, L/c; the revolution time, T_0; the period of the perturbing electromagnetic field, T_p; the period of synchrotron oscillations, T_s; the instability rise time τ_i. In all the work done we always assume $\tau_i > T_0$.

We can consider two cases:

$$\text{I) slow instabilities: } \tau_i > T_s > T_0 \tag{1.1}$$

$$\text{II) fast instabilities: } T_i > \tau_i > T_0 \tag{1.2}$$

Cases I) and II) can be divided in two subcases:

$$\text{a) long wavelength: } \frac{L}{cT_p} << 1 \tag{1.3}$$

$$\text{b) short wavelength: } \frac{L}{cT_p} >> 1 \tag{1.4}$$

For each of these cases it is possible to write an approximate expression for the stability condition or threshold current.

Case Ia describes the coupled bunch instabilities and will be discussed in Section 5; case Ib in Section 6; case IIb, describing what is usually called the microwave instability or bunch lengthening effect, in Section 7, together with IIa.

It is of course possible to think of instability mechanisms different from the beam wall interaction considered here. However in the range of particle densities obtained in circular accelerator and storage rings, these effects have not been observed and will not be discussed here.

II. General Formalism

We consider a beam with B bunches and write the equations of synchrotron oscillations in the form[4]

$$\dot{\phi}_p(t) = \eta\,\omega_0\,\varepsilon_p(t) \qquad p = 0, 1, \ldots, B-1 \tag{2.1}$$

$$\dot{\varepsilon}_p(t) = -\frac{\omega_0\,\nu_0^2}{\eta}\,\phi_p(t) + \xi_p(t) \tag{2.2}$$

where ϕ_p is the phase, for a particle in the p-th bunch, relative to a synchronous particle moving with constant energy, E_s, and constant angular velocity $\omega_0 = 2\pi/T_0$; ε_p is the relative energy deviation, $\varepsilon_p = (E_p-E_s)/E_s$; ν_0 is the small amplitude synchrotron oscillation frequency; η is the frequency slip factor; $\xi_p(t)$ describes the additional energy loss due to the interaction with the environment, $\bar{\xi}(t)$, and to the nonlinear part of the rf voltage, $\xi_{NL}(t)$.

$$\xi(t) = \bar{\xi}(t) + \xi_{NL}(t) \tag{2.3}$$

This last part is included to be able to consider the effect of Landau damping on beam stability.

The additional energy loss can be related to the beam current $J_{||}(\theta, t)$, $\theta = \omega_0 t + \phi$ being the azimuthal angle. If $Z(\omega)$ is the impedance of the beam environment, the current J will produce a longitudinal electric field[3]

$$E_{||}(\theta,t) = -\frac{1}{2\pi R}\int d\omega Z(\omega) J_{||}(\theta,\omega) e^{i\omega t} \tag{2.4}$$

where R is the average ring radius and

$$J_{||}(\theta,\omega) = \frac{1}{2\pi}\int dt\, J_{||}(\theta,t) e^{-i\omega t} \tag{2.5}$$

The relative energy loss per revolution of particle ℓ, whose azimuthal position is $\theta_\ell(t)$, during the n-th revolution can be written as

$$\xi_\ell(n) = \frac{1}{ET_0}\int_{nT_0}^{(n+1)T_0} e E_{||}(\theta_\ell(t),t) v dt \tag{2.6}$$

or, using (2.4), as

$$\xi_\ell(n) = \frac{e\omega_0^2}{(2\pi)^2 E}\int d\omega Z(\omega)\int_{nT_0}^{(n+1)T_0} J_{||}(\theta_\ell(t),\omega) e^{i\omega t} dt \tag{2.7}$$

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When $\nu_0 << 1$ it is a good approximation to take $\bar{\xi}_\ell(t) = \bar{\xi}_\ell(nT)$ and use this expression in (2.2).

Equations (2.1) and (2.2) can be derived from an Hamiltonian

$$H = \sum_{n=0}^{B-1} \left\{\frac{1}{2}\eta\omega_0\varepsilon_n^2 + \frac{1}{2}\frac{\omega_0\nu_0^2}{\eta}\phi_n^2\right\} + U + U_{NL} \tag{2.8}$$

where

$$\bar{\xi}_p = -\frac{\partial U}{\partial \phi_p} \tag{2.9}$$

and U_{NL} describes the nonlinear part of the rf force.

III. Slow Instabilities

To study the stability properties we use the linearized Vlasov equation. For the slow instabilities it is convenient to use instead of the variables ψ,ε the variables I,ψ defined by the canonical transformation function

$$F = -\frac{1}{2}\frac{\nu_0}{\eta}\sum_{n=0}^{B-1}\phi_n^2 \, tg\,(\nu_0\omega_0 t + \psi_n) \tag{3.1}$$

From (3.1) we obtain

$$\phi_n = \left(\frac{2\eta\ I_n}{\nu_0}\right)^{1/2} \cos\,(\nu_0\omega_0 t + \psi_n) \tag{3.2}$$

$$\varepsilon_n = \left(\frac{2\ \nu_0\ I_n}{\eta}\right)^{1/2} \sin\,(\nu_0\omega_0 t + \psi_n) \tag{3.3}$$

The new Hamiltonian is

$$\bar{H} = \bar{U} + \bar{U}_{NL} \tag{3.4}$$

where the term $\bar{U}_{NL}$ can be written as

$$\bar{U}_{NL} = -\frac{1}{16}\eta\,\omega_0\,B^2\sum_{n=0}^{B-1} I_n^2 \tag{3.5}$$

and U is obtained from (2.9) with ϕ_p given by (3.2).

To evaluate $\bar{U}$ we write the single particle current as

$$j_{||}^{(p)}\ (\theta,t) = e\omega_0 \sum_{k=-\infty}^{+\infty}\delta(\theta-\omega_0 t-2\pi k-2\pi p/B-\phi_p) \tag{3.6}$$

or, using the Poisson sum rule and a Bessel function expansion, as

$$j_{||}^{(p)}\ (\theta,t) = 2\pi e\,\omega_0 \sum_{k,m=-\infty}^{+\infty}(-i)^m J_m[k\ r_p]. \\ .e^{ik(\theta - \omega_0 t - 2\pi p/B) + im\,(\nu_0\omega_0 t + \psi_p)} \tag{3.7}$$

where we have used the notations

$$r_p = (2\eta\ I_p/\nu_0)^{1/2} \tag{3.8}$$

We now introduce the distribution function $f^{(p)}(I_p, \psi_p, t)$ for particles in the p-th bunch and use it to evaluate the total bunch current.

Assuming

$$f^{(p)}(I_p, \psi_p, t) = f^{(p)}\ (I_p,\ \psi_p)e^{i\Omega t} \tag{3.9}$$

we obtain

$$J_{||}(\theta,t) = 2\pi e\omega_0 \sum_{k,m} e^{ik(\theta-\omega_0 t)+i(\Omega+m\nu_0\omega_0)t} \\ \cdot e^{-2\pi ikp/B}(-i)^m\int_0^\infty dI_p \int_0^{2\pi} d\psi_p f^{(p)}(I_p,\psi_p)J_m\left(k\sqrt{\frac{2\eta I_p}{\nu_0}}\right)e^{im\psi_p} \tag{3.10}$$

From (3.10), (2.7), (2.9) we obtain for the self-potential $\bar{U}$

$$\bar{U} = \frac{e^2\omega_0^2}{(2\pi)^2 E}\sum_{n,p=0}^{B-1}\sum_{k=-\infty}^{+\infty}\frac{Z(\Omega+s\nu_0\omega_0-k\omega_0)}{ik}\ e^{2\pi ik(p-n)/B} \\ \cdot \sum_{r,s=-\infty}^{+\infty} i^{r+s}(-1)^s J_r\left(k\sqrt{\frac{2\eta I_p}{\nu_0}}\right)e^{i[(r+s)\nu_0\omega_0+\Omega]t} \\ e^{ir\psi_0}\int_0^\infty dI_n\int_0^{2\pi} d\psi_n\ e^{is\psi_n}\ J_s\left(k\sqrt{\frac{2\eta I_n}{\nu_0}}\right)f^{(n)}(I_n,\psi_n) \tag{3.11}$$

Under the assumption

$$\Omega << \nu_0\omega_0 \tag{3.12}$$

we can use a slowly varying phase and amplitude approximation and neglect all terms except the slowly oscillating ones, for which r+s = o.[5]

The approximate Hamiltonian is

$$\bar{U} = \frac{e^2\omega_0^2}{(2\pi)^2 E}\sum_{n,p=0}^{B-1}\sum_{k,r=-\infty}^{+\infty}\frac{Z(\Omega-k\omega_0-r\nu_0\omega_0)}{ik}\ e^{2\pi ik(p-n/B)} \\ J_r\left(k\sqrt{\frac{2\eta I_p}{\nu_0}}\right)e^{ir\psi_p}\int_0^\infty dI_n\int_0^{2\pi} d\psi_n f^{(n)}(I_n,\psi_n)e^{i\Omega t-ir\psi_n} \\ J_r\left(k\sqrt{\frac{2\eta I_n}{\nu_0}}\right) \tag{3.13}$$

and this will be used to describe the slow instabilities.

We can now write the Vlasov equation for $f^{(p)}$ as

$$\frac{\partial f^{(p)}}{\partial t} + \frac{\partial(\bar{U}+U_{NL})}{\partial I_p}\frac{\partial f^{(p)}}{\partial \psi_p} - \frac{\partial U}{\partial \psi_p}\frac{\partial f^{(p)}}{\partial I_p} = 0 \tag{3.14}$$

$$p = 0,\ \ldots,\ B-1$$

We look for a solution of (3.14) of the form

$$f^{(p)}(I_p,\psi_p) = f_0(I_p) + f_q^{(p)}(I_p)e^{iq\psi_p + i\Omega t} \tag{3.15}$$

Since f_0 is the same for all bunches we are limiting ourselves to the case where the unperturbed bunches are all equal.

Using (3.15) we can redefine $\bar{H}$ as

$$\bar{H} = \bar{U}^{(0)} + \bar{U}^{(1)} + \bar{U}_{NL} = \bar{U}^{(1)} + U^* \tag{3.16}$$

where $\bar{U}^{(0)}$ and $\bar{U}^{(1)}$ are obtained from (3.11) with f equal to f_0 or f_q.

We can see by inspection of (3.14) that any function $f_o(I)$ is a solution describing an equilibrium state. The function $f_q^{(p)}$, describing a small perturbation around this equilibrium, must again satisfy (3.14). Neglecting second order term in $f_q^{(p)}$ this equation can be written as:

$$f_q^{(p)}(I_p) = \frac{qe^2\omega_o^2}{2\pi E}\frac{(\partial f_o/\partial I_p)}{\Omega+q(\partial U^*/\partial I_p)} \quad . \tag{3.17}$$

$$\cdot \sum_{n=o}^{B-1}\sum_{k=-\infty}^{+\infty} \frac{Z(\Omega-q\nu_o\omega_o-k\omega_o)}{ik} e^{2\pi ik(p-n)/B} J_q\left(k\sqrt{\frac{2\eta I_p}{\nu_o}}\right)\star\sigma_q^{(n)}(k)$$

where

$$\sigma_q^{(n)}(k) = \int_0^\infty dI_n\, f_q^{(n)}(I_n)\, J_q\left(k\sqrt{\frac{2\eta I_n}{\nu_o}}\right) \tag{3.18}$$

It is interesting to note that with the choice $f_q \approx e^{iq\psi}$ we have obtained an equation, (3.17), independent of ψ and in which the different q modes are not coupled. We will call these modes the "synchrotron modes" and will refer to the index K as the index of "azimuthal modes".

Considering (3.17) as a matrix equation for the vector $f_q^{(p)}$, p = 0, ..., B-1 one can see that this matrix is a cyclic matrix in p,n and that its solutions are of the form

$$f_q^{(p)} = f_{q,s}\, e^{2\pi isp/B} \quad , s=o, \ldots, B-1 \tag{3.19}$$

The index "s" defines the coupled bunch modes. For a given s the bunches oscillate with a phase difference 2π s/B.

Using (3.19), equation (3.17) becomes

$$f_{q,s}(I) = \frac{qe^2\omega_o^2B}{2\pi E}\frac{\partial f_o/\partial I}{\Omega+q(\partial U^*/\partial I)} \tag{3.20}$$

$$\sum_{k=-\infty}^{+\infty} \frac{Z(\Omega-\nu_o\omega_o-s\omega_o+kB\omega_o)}{i(s-kB)} J_q\left[(s-kB)\sqrt{\frac{2\eta I}{\nu_o}}\right]\sigma_{q,s}(s-kB)$$

with

$$\sigma_{q,s}(s-kB) = \int_0^\infty dI\, f_{q,s}(I)\, J_q\left[(s-kB)\sqrt{\frac{2\eta I}{\nu_o}}\right] \tag{3.21}$$

equations which have already been derived and discussed by many authors[5].

Our problem is now reduced to finding the eigenvalues and eigenvectors of (3.20). We have no general solution of these equations, although solutions have been found for particular forms of the impedance $Z(\omega)$ or by introducing other approximations. In the next sections we will discuss some of these solutions.

IV. The Impedance

It is important to notice that the solutions of (3.20) depend on the form of the impedance function $Z(\omega)$. Although it is difficult to know or to prescribe the impedance function of a storage ring, we can make some qualitative remarks based on results obtained in existing rings.

The major terms contributing to $Z(\omega)$ are:

1. The impedance of a perfectly conducting wall of uniform cross section; this produces a force proportional to $\partial\lambda/\partial\theta$ or $Z(\omega) \approx i\omega$;[6]

2. The impedance due to the resistivity of the wall;[7]

3. A broad band impedance describing the effect of small vacuum chamber discontinuities, electrodes, small cavity-like devices, etc.; this can be simulated by a Q = 1 resonator with a resonant frequency near to the vacuum pipe cut-off frequency, ω_{co};[8] this description is good at ω of the order of ω_{co} but breaks down at very low frequencies;

4. Low frequency resonant impedance produced by parasitic modes in the radio frequency accelerating cavities; this can be important for $\omega < \omega_{co}$ and can have large Q and a high impedance peak value.[2]

The term in 1 does not produce instabilities and is small for highly relativistic beams. The resistive wall impedance is smaller than the broad band impedance at high frequency and can be smaller than the resonant impedance at low frequency. In what follows we will neglect terms 1 and 2, for simplicity.

One major difference between the broad band and the resonant impedance is that a signal at frequency ω, induced in the broad band impedance decays in a time of the order of the $1/\omega$, while one induced in the resonant impedance decays over a much longer time. In terms of the impedance decay time $1/\Gamma$ and the bunch separation T_o/B we can say that

$$\frac{1}{\Gamma} > \frac{T_o}{B} \qquad \text{for resonant impedance,}$$

$$\frac{1}{\Gamma} < \frac{T_o}{B} \qquad \text{for broad-band impedance.}$$

This condition indicates that the coupled bunch instabilities should be dominated by the resonant impedance and that single bunch instabilities should be dominated by the broad band impedance.

V. Coupled Bunch Instability

Using the conditions (1.1), (1.3), discussed in the previous sections, i.e., $\omega L/c \ll 1$, $1/\tau_i < \nu_o\omega_o$, we can simplify (3.20) by expanding the Bessel functions, for small values of the argument, as

$$J_q(Z) \approx \frac{1}{q!}\left(\frac{Z}{2}\right)^q \tag{5.1}$$

An examination of this simplified equation shows that we can obtain a solution of the form[9]

$$f_{q,s}(I) = M\frac{\partial f_o/\partial I}{\Omega+q(\partial U^*/\partial I)}\left(\frac{\eta I}{2\nu_o}\right)^{q/2} \tag{5.2}$$

M being a normalization constant. The corresponding eigenvalue is a solution of the dispersion integral

$$1 = -i \frac{e^2 \omega_0^2 B}{2\pi E(q-1)!} \sum_{k=-\infty}^{+\infty} Z[(kB-s-q\nu_0)\omega_0] (s-kB)^{q-1} \cdot \tag{5.3}$$

$$\cdot \int_0^\infty dI \frac{\partial f_0/\partial I}{\Omega+q(\partial U/\partial I)} \left(\frac{\eta I}{2\nu_0}\right)^{q/2} J_q\left[(s-kB)\sqrt{\frac{2\eta I}{\nu_0}}\right]$$

In the absence of Landau damping $[q(\partial U^*/\partial I) << \Omega]$ and for a Gaussian unperturbed bunch of rms length σ_ϕ,

$$f_0(I) = \frac{N\eta}{2\pi\,\nu_0\,\sigma_\phi^2}\, e^{-(\eta I/\nu_0\sigma_\phi^2)} \tag{5.4}$$

we obtain from (5.3) for the collective oscillation frequency of the s-th mode of the coupled bunches and the q-th synchrotron mode

$$\frac{\Omega_q^{(s)}}{\omega_0} = \frac{e\,I_0\,B\eta\,Z^{(s)}_{eff,}}{2\pi\,\nu_0 E}\;\frac{\sigma_\phi^{2q-2}}{2^q(q-1)!} \tag{5.5}$$

where $I_0 = e\,\omega_0 N/2\pi$ is the average current per bunch and the effective coupling impedance is

$$Z^{(s)}_{eff,q} = \sum_{k=-\infty}^{+\infty} iZ\left[(kB-s-q\,\nu_0)\omega_0\right](s-kB)^{2q-1}\, e^{-(s-KB)^2\sigma_\phi^2/2} \tag{5.6}$$

This equation was obtained by several authors[10] to explain the first observations of coupled bunch instabilities and has been widely discussed in the literature.[11]

VI. The High Frequency Slow Instability

As discussed in Section 4 we can now consider a single bunch and an impedance as the broad band impedance. Neglecting the bunch index we have from (3.17)

$$f_q(I) = -\frac{qe^2\,\omega_0^2}{2\pi E}\,\frac{\partial f_0/\partial I}{\Omega+q(\partial U^*/\partial I)} \sum_k \frac{Z[(k-q\,\nu_0)\omega_0+\Omega)]}{ik} * \tag{6.1}$$

$$* \, J_q\left(k\sqrt{\frac{2\eta I}{\nu_0}}\right)\sigma_q(k)$$

$$\sigma_q(k) = \int_0^\infty dI\, f_q(I)\, J_q\left(k\sqrt{\frac{2\eta I}{\nu_0}}\right) \tag{6.2}$$

We will discuss (6.1) in the case of no Landau damping, i.e., $\partial U^*/\partial I = 0$.

Equation (6.1) has been obtained and studied in great detail by Sacherer[12] and we will follow his work.

Let us introduce a new function

$$u^{(q)}(I) = \left(\frac{\partial f_0}{\partial I}\right)^{-1/2} f_q(I) \tag{6.3}$$

and rewrite (6.1) explicitly in the form of an integral equation

$$u^{(q)}(I) = \frac{-qe^2\omega_0^2}{2\pi E\Omega}\int_0^\infty dI'\, u^{(q)}(I')\, G(I,I') \tag{6.4}$$

with

$$G(I,I') = \sum_{k=1}^{\infty} \frac{Z(k-q\nu_0) - Z^*(k+q\nu_0)}{ik} \times \tag{6.5}$$

$$\times \left(\frac{\partial f_0}{\partial I}\right)^{1/2} J_q\left(k\sqrt{\frac{2\eta I}{\nu_0}}\right)\left(\frac{\partial f_0}{\partial I'}\right)^{1/2} J_q\left(k\sqrt{\frac{2\eta I'}{\nu_0}}\right)$$

To write the symmetric Kernel $G(I,I')$ in the form (6.5) we have also made use of the property $Z(-\omega) = Z^*(\omega)$ (* = complex conjugate) which follows from causality. If the quantity $[Z(k-q\nu_0) - Z^*(k+q\nu_0)]/ik$ is real the eigenvalues of (6.4) are also real. This is true for the case of a perfectly conducting wall where $Z(\omega)/i\omega = L_\omega$ is a constant.[6]

Let $F_m^{(q)}(I)$ be a complete set of eigenfunctions of the integral equation (6.4),

$$F_m^{(q)}(I) = \frac{-qe^2\omega_0^2}{2\pi E\Omega_{q,m}} \sum_k \frac{Z(k-q\nu_0)}{ik} \left(\frac{\partial f_0}{\partial I}\right)^{1/2} J_q\left(k\sqrt{\frac{2\eta I}{\nu_0}}\right) \cdot \tag{6.6}$$

$$\cdot \int_0^\infty dI'\, F_m^{(q)}(I')\left(\frac{\partial f_0}{\partial I'}\right)^{1/2} J_q\left(k\sqrt{\frac{2\eta I}{\nu_0}}\right)$$

satisfying the orthonormality condition

$$\int_0^\infty F_m^{(q)}(I)\, F_p^{(q)}(I)\, dI = \delta_{m,p} \tag{6.7}$$

We can now expand both $u^{(q)}(I)$ and $\sigma_q(k)$ in terms of the eigenfunctions

$$u^{(q)}(I) = \sum_m D_{q,m} F_m^{(q)}(I) \tag{6.8}$$

$$\sigma_q(k) = \sum_m E_{q,m}\,\Lambda^{(q)}_{m,k} \tag{6.9}$$

with

$$\Lambda^{(q)}_{m,k} = \int_0^\infty dI\, F_m^{(q)}(I)\left(\frac{\partial f_0}{\partial I}\right)^{1/2} J_q\left(k\sqrt{\frac{2\eta I}{\nu_0}}\right) \tag{6.10}$$

The eigenvalues of (6.1) can be written in terms of the $\Omega^{(q)}_{m,k}$ as

$$\Omega_{q,m} = -\frac{qe^2\omega_0^2}{2\pi E}\sum_{k=-\infty}^{+\infty} \frac{Z(k-q\nu_0)}{ik}\left[\Lambda^{(q)}_{m,k}\right]^2 \tag{6.11}$$

The eigenvalues and eigenfunctions of (6.4) depend on f_0 and an $Z(\omega)$. We do not have any general solution of (6.4). A solution has been found, however, for the case of a parabolic charge distribution and $Z(\omega)/i\omega = L_\omega$[13]. Although this case "per se" is not very interesting, all the eigenvalues are real and the beam is stable, it can be used to evaluate the complex eigenvalues when the impedance is of the form

$$\frac{Z(\omega)}{i\omega} = L_\omega + \frac{Z_1(\omega)}{i\omega} \tag{6.12}$$

with

$$\left|\frac{Z_1(\omega)}{i\omega}\right| << L_\omega \tag{6.13}$$

When (6.12), (6.13) are satisfied we can use perturbation methods to write the eigenfunctions and the eigenvalues as

$$F_m^{(q)}(I) = F_m^{(q)(o)}(I) + F_m^{(q)(1)}(I) \tag{6.14}$$

$$\Omega_{q,m} = \Omega^{(o)}_{q,m} + \Omega^{(1)}_{q,m} \tag{6.15}$$

with $\Omega^{(o)}_{q,m}$ obtained for $Z_1 = 0$ and

$$\Omega^{(1)}_{q,m} = -\frac{qe^2\omega_o^2}{2\pi E} \sum_{k=1}^{\infty} \frac{Z_1(k-q\nu_o) - Z_1^*(k+q\nu_o)}{ik} \left[\Lambda^{(q)(o)}_{m,k}\right] \tag{6.16}$$

Let us now find the zeroth order solution. We assume

$$f_o(I) = \frac{3}{4\pi} \frac{2\eta N}{\nu_o\phi_L^2} \left\{1 - \frac{2\eta I}{\nu_o\phi_L^2}\right\}^{1/2} \tag{6.17}$$

corresponding to a longitudinal density

$$\lambda_o(\phi) = \frac{3N}{4\phi_L}\left\{1 - \frac{\phi^2}{\phi_L^2}\right\} \tag{6.18}$$

With this choice of $f_o(I)$ one has[13]

$$\Lambda^{(q)(o)}_m(k) = B_{q,m} \frac{j_{m+1/2}(k\phi_L)}{(k\phi_L)^{1/2}}, m=q+2,q+4\ldots, \tag{6.19}$$

with

$$B^2_{q,m} = \frac{3}{8}\frac{N\eta}{\nu_o}(2m+1)\frac{(m+q-1)!!\ (m-q-1)!!}{(m+q)!!\ (m-q)!!} \tag{6.20}$$

Introducing the average current $I_o = e\omega_o N/2\pi$ and an "effective impedance"

$$Z^{(o)}_{eff} = 3\pi\frac{\omega_o L_\omega}{\phi_L^2} \tag{6.21}$$

we can write the collective frequency as

$$\Omega^{(o)}_{q,m} = \frac{e\ I_o Z^{(o)}_{eff}\ \eta\ \omega_o}{2\pi\ E\ \nu_o}\ q\ \frac{(m+q-1)!!\ (m-q-1)!!}{(m+q)!!\ (m-q)!!} \tag{6.22}$$

Similarly we can write $\Omega^{(1)}$ as

$$\Omega^{(1)}_{q,m} = \frac{eI_o Z^{(1)}_{eff}\ \eta\ \omega_o}{2\pi\ \nu_o\ E}\ q\ \frac{(m+q-1)!!\ (m-q-1)!!}{(m+q)!!\ (m-q)!!} \tag{6.23}$$

with

$$Z^{(1)}_{eff} = -\frac{3i}{8}(2m+1)\int_{-\infty}^{+\infty}\left[dy\, Z_1\left(y - q\nu_o\phi_L\right)\frac{\omega_o}{\phi_L}\right]\frac{\left[J_{m+1/2}(y)\right]^2}{y^2} \tag{6.24}$$

The formulae (6.21), (6.24) can be used to evaluate the coupling impedance for a single bunch slow slow instability. It must be remembered, however, that (6.24) can only be applied for an impedance of the form (6.12), (6.13) and that for an arbitrary impedance the result (6.19) is not generally valid. In particular, since the term L_ω in (6.12) produced by smooth conducting walls is very small for high energy beams, the formula (6.24) can only be approximately applied in frequency regions where the broad band impedance is slowly varying.

VII. The High Frequency Fast Instability

When the instability rise time is shorter than the synchrotron period we cannot apply the results of the previous sections, because in deriving the Hamiltonian (3.13) we used the approximation $\Omega \ll \nu_o\,\omega_o$. A different approach can be used when $\Omega \gg \nu_o\omega_o$[14]. In this case the "synchrotron modes" q are all coupled while using the Hamiltonian (3.13) they are uncoupled as shown by the form (3.15) of the distribution function and equation (3.17).

The coupling of all synchrotron modes plays an important role in the fast instability. To take it into account it is convenient to use instead of the canonical transformation (3.1) the transformation

$$F = -\frac{1}{2}\frac{\nu_o}{\eta}\phi^2\ \mathrm{tg}\ \delta \tag{7.1}$$

giving

$$\phi = \left(\frac{2\eta I}{\nu_o}\right)^{1/2}\cos\delta \tag{7.2}$$

$$\varepsilon = -\left(\frac{2\ \nu_o I}{\eta}\right)^{1/2}\sin\delta \tag{7.3}$$

We limit ourselves to consider a single bunch, which is the only relevant case.

Following the same procedure as in Section 3 we obtain the Hamiltonian

$$H = \nu_o\ \omega_o\ I + \overline{U} \tag{7.4}$$

with

$$\overline{U} = \frac{e^2\omega_o^2}{8\pi^2 E}\sum_{k=-\infty}^{+\infty}\frac{Z(k\omega_o - \Omega)}{ik} \cdot \int_0^\infty dI' \int_0^{2\pi} d\delta' f(I',\delta') e^{ik[\phi(I,\delta)-\phi(I',\delta')]} \tag{7.5}$$

Assuming

$$f = f_o(I) + f_1(I,\delta)e^{i\nu_o\omega_o\Omega t} \tag{7.6}$$

and linearizing the Vlasov equation we have

$$\nu_o\omega_o\left(i\Omega f_1 + \frac{\partial f_1}{\partial\delta}\right) = \frac{\partial\overline{U}^{(1)}}{\partial\delta}\frac{\partial f_o}{\partial I} \tag{7.7}$$

where $\overline{U}^{(1)}$ is obtained from (7.5) for $f = f_1$.

The solution of (7.7) can be written, using the periodicity in δ, as

$$f_1(I,\delta) = -\frac{1}{\nu_o\omega_o}\frac{\partial f_o}{\partial I}\frac{e^{-i\Omega\delta}}{e^{2\pi i\Omega}-1}\int_\delta^{\delta+2\pi} d\delta' \frac{\partial U^{(1)}(I,\delta')}{\partial\delta'} e^{i\Omega\delta'} \tag{7.8}$$

Defining

$$\Lambda_m = \int_0^\infty dI \int_0^{2\pi} d\delta\ f_1(I,\delta)e^{-im\phi(I,\delta)} \tag{7.9}$$

we can reduce the integral equation (7.8) to a linear algebraic system

$$\Lambda_m = \sum_{n=-\infty}^{+\infty} T_{m,n}\ \Lambda_n \tag{7.10}$$

with

$$T_{m,n} = -\frac{e^2\ \omega_o}{8\pi^2 E\nu_o}\frac{Z(n\omega_o - \nu_o\omega_o\Omega)}{e^{2\pi i\Omega}-1}\int_0^\infty dI\ \frac{\partial f_o}{\partial I}\left(\frac{2\eta I}{\nu_o}\right)^{1/2} \cdot \int_0^{2\pi} d\delta d\delta'\ \sin(\delta+\delta')e^{i\Omega\delta' + in\phi(I,\delta+\delta') - im\phi(I,\delta)} \tag{7.11}$$

The solutions of our problem are the values of Ω for which the equation

$$\det\left|\delta_{m,n} - T_{m,n}\right| = o \tag{7.12}$$

is satisfied.

We emphasize here that, contrary to the coasting beam case for a bunched beam all frequency components $n\omega_0$ are coupled. Also all synchrotron modes $f_1(I,\delta) \approx f_1(I)\, e^{iq\delta}$ are coupled, contrary to the slow instability case. In this coupling of all modes lies the difficulty of solving the fast blow-up bunched beam case.

To study equation (7.10) we use the following approach. We consider only high frequency perturbations, i.e., the case

$$n\,\sigma_\phi \gtrsim 1 \tag{7.13}$$

where σ_ϕ is the rms bunch length in units of machine radius. Then we consider the fast blow-up case

$$\omega_0 >> |\Omega|\nu_0\omega_0 >> \nu_0\omega_0 \tag{7.14}$$

Using these two conditions we simplify the expression of the matrix element (7.11) by assuming $Z(n\omega_0 - \Omega\nu_0\omega_0) \approx Z(n\omega_0)$ and by evaluating the integral over δ in the limit $I_m\Omega \to -\infty$. We also rewrite the integration over I and δ as an integral over ε and ϕ to obtain

$$T_{m,n} \approx -\frac{ie^2\omega_0}{8\pi^2 E\nu_0} Z(n\omega_0) \int d\varepsilon d\phi\; e^{i(n-m)\phi} \frac{\partial f_0/\partial\varepsilon}{\Omega-(n\eta/\nu_0)\varepsilon} \tag{7.15}$$

We now restrict ourselves to the case of a Gaussian bunch

$$f_0 = N\,\lambda(\phi)\,G(\varepsilon) \tag{7.16}$$

$$\lambda(\phi) = \frac{1}{\sqrt{2\pi}\;\sigma_\phi} e^{-\phi^2/2\sigma_\phi^2} \tag{7.17}$$

$$G(\varepsilon) = \frac{1}{\sqrt{2\pi}\;\sigma_\varepsilon} e^{-\varepsilon^2/2\sigma_\varepsilon^2} \tag{7.18}$$

We also define

$$\lambda_n = \frac{1}{2\pi}\int d\phi\; \lambda(\phi) e^{in\phi} = \frac{1}{2\pi} e^{-n^2\sigma_\phi^2/2} \tag{7.19}$$

The matrix element can now be written as

$$T_{m,n} = -\frac{ie^2\omega_0 N}{4\pi^2 E n\nu_0} \frac{Z(n\omega_0)}{n} \lambda_{n-m} \int d\varepsilon\; \frac{\partial G/\partial\varepsilon}{(\Omega/n\eta)-\varepsilon} \tag{7.20}$$

or, using (7.18) as

$$T_{m,n} = -\frac{ie^2\omega_0}{2\pi\eta}\frac{N}{E\,\sigma_\varepsilon^2} \frac{Z(n\omega_0)}{n} \lambda_{n-m}\, h\left(\frac{\Omega}{n\sigma_\phi}\right) \tag{7.21}$$

with

$$h(y) = \int_0^{+\infty} dx\; x e^{-x^2/2-ixy} \tag{7.22}$$

This function has the property that

$$\begin{aligned} &h(o) = 1 \\ &|h(y)| \leq 1 \quad \text{if} \quad y \neq o\,, \quad \text{Im}\, y < o \end{aligned} \tag{7.23}$$

so that it has a maximum for y = 0 and decreases when y increases.

It is also interesting to notice that the matrix element (7.20) or (7.21) can be also used to describe the limiting case of a coasting beam by taking the other limit $\sigma_\phi \to \infty$ or

$$\lambda_{n-m} = \delta_{n,m} \tag{7.24}$$

In this case the matrix T is diagonal and we obtain the usual coasting beam dispersion relation(3)

$$1 = -i\frac{eI_0}{2\pi E\eta}\frac{Z(n\omega_0)}{n}\int d\varepsilon\; \frac{\partial G/\partial\varepsilon}{(\Omega/n\eta)-\varepsilon} \tag{7.25}$$

For a bunched beam we have to solve the more complicated equation (7.10) with $T_{m,n}$ given by (7.21). We can obtain a simple but approximate solution if we make the approximation that the impedance is large only near a frequency n_0 and is nearly constant over the range $n_0 - N_b$, $n_0 + N_b$ where $N_b \sim 1/\sigma_\phi$ defines the range over which the bunch form factor λ_m remains near to one. In this simple model, we can rewrite (7.21) as

$$T^*_{m,n} \approx \frac{ieI_0}{2\pi E\eta\;\sigma_\varepsilon^2} \frac{Z(n_0\omega_0)}{n_0}\, h\left(\frac{\Omega}{n_0\sigma_\phi}\right) e^{-(n-m)^2\sigma_\phi^2/2} \tag{7.26}$$

The eigenvectors of the matrix $T^*_{m,n}$ are of the form

$$\Lambda_m = e^{i\alpha m} \tag{7.27}$$

and the corresponding eigenvalue is obtained from

$$1 = \frac{ieI_0}{2\pi E\eta\sigma_\varepsilon^2}\frac{Z(n_0\omega_0)}{n_0}\, h\left(\frac{\Omega(\alpha)}{n_0\sigma_\phi}\right)\sum_{n=-\infty}^{+\infty} e^{-n^2\sigma_\phi^2/2+in\alpha} \tag{7.28}$$

Since $h(x) \leq 1$ and the sum over n has a maximum for $\alpha = 0$ we obtain a solution of (7.29) only if

$$\frac{eI_0}{2\pi E\eta\sigma_\varepsilon^2}\sum_{n=-\infty}^{+\infty} e^{-n^2\sigma_\phi^2/2}\left|\frac{Z(n_0\omega_0)}{n_0}\right| \geqslant 1 \tag{7.29}$$

If this condition is not satisfied we contradict our initial assumption $|\Omega| > 1$. On the other hand if we satisfy (7.29) we can find a solution of (7.28) with $|\Omega|>1$, consistent with the way we have derived (7.20). This means that we can have a fast blow-up only if the beam current is such as to satisfy (7.29) and that at lower beam current there is no possibility of a fast blow-up.

It is also interesting to notice that for a given $\Omega/\nu_0\omega_0$, the function $h(\Omega/n_0\sigma_\phi)$ is larger when $n_0\sigma_\phi > |\Omega| > 1$, so that a high frequency impedance is more effective in producing a fast blow-up.

VIII. Conclusions

We have reviewed the general problem of the longitudinal stability of bunched beams. Although there is no general solution it is possible to identify regions in the frequency-risetime space where we can obtain approximate solutions. The collective oscillation frequency can be written in the form

$$\frac{\Omega}{\nu_0\omega_0} = \frac{eI_0}{2\pi\nu_0 E\eta} Z_{eff}$$

and expressions for the effective coupling impedance are given for the high or low frequency and slow and fast blow-up regimes.

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