

CHANGE OF THE ENERGY DISTRIBUTION IN AN ELECTRON STORAGE RING BY A DIPOLE-OCTUPOLE WIGGLER

A. Hofmann, J. Jowett and S. Myers
CERN, 1211 Geneva 23, Switzerland

Summary

A dipole-octupole wiggler is proposed to change the energy distribution of the electrons in a storage ring so that the peak current and the related collective effects are reduced. With a dipole-quadrupole wiggler the radiation damping partition can be changed and with a dipole-octupole wiggler this partition becomes dependent on the phase oscillation amplitude. These wigglers can be operated so that the longitudinal damping is negative for small amplitudes but increases for larger amplitudes. The particle distribution will peak around the amplitude which gives neither damping nor antidamping. This leads to a longitudinal blow-up of the bunch core without affecting very much the tails or the quantum lifetime. The distribution has been calculated analytically with an appropriate Fokker-Planck equation and by computer simulation. The results allow the optimization of the wiggler parameters. Due to the sum rule for the damping partitions this wiggler also affects the distribution in horizontal betatron amplitudes.

1. Introduction

Very large electron-positron storage rings such as LEP¹ have bunches which are very short compared to the circumference. This can lead to strong single-bunch effects such as bunch lengthening and coherent betatron tune shifts. Increasing the bunch length and thereby decreasing the peak current improves this situation. To stay within the short RF wavelength and the limited energy acceptance of the machines an efficient bunch lengthener should affect mainly the core and to a lesser extent the tails of the distribution.

A combined function dipole-quadrupole wiggler² changes the damping partition but affects the core and the tails of a bunch the same way. However a combined function dipole-octupole wiggler introduces a dependence of the damping partition on the phase oscillation amplitude³. With both wigglers the longitudinal damping can be weak or even negative for small amplitudes but growing with increasing amplitudes⁴. This depopulates the small amplitudes and makes the tails steeper, leading to the desired distribution.

2. Longitudinal Particle Motion
Neglecting Quantum Excitation

The energy oscillation of an ultra-relativistic electron in a storage ring is described by the equation⁵ (neglecting non-linearities of the RF wave form):

$$\ddot{\epsilon} + \frac{\omega_0}{2\pi} \frac{1}{E_0} \frac{dU}{d\epsilon} \dot{\epsilon} + \Omega_s^2 \epsilon = 0 \quad (1)$$

with $\epsilon = \Delta E/E_0 =$ relative deviation from the equilibrium energy E_0 , $\omega_0 =$ revolution frequency, $\Omega_s =$ synchrotron frequency, $U =$ energy loss per revolution of one electron due to synchrotron radiation. In a dipole-octupole wiggler the dependence of the magnetic field B_w on the transverse horizontal distance x_0 from the equilibrium orbit is

$$B_w = B_2 + \frac{1}{6} \frac{\partial^3 B_0}{\partial x^3} x_0^3 = \frac{E_0}{ec} \left(\frac{1}{\rho_w} + \frac{1}{6} K'' x_0^3 \right)$$

where ρ_w is the bending radius due to the dipole field and K'' the octupole strength parameter for the equilibrium energy E_0 . The energy loss U_w in this wiggler of length L_w becomes

$$U_w = \frac{2}{3} \frac{r_e L_w E_0^2 E^2}{(m_0 c^2)^3} \left(\frac{1}{\rho_w} + \frac{1}{6} K'' x_0^3 \right)^2 \quad (2)$$

In this and the next chapter it is assumed that in the wiggler the horizontal beta function is small but the dispersion D large so that the horizontal excursion x_0 is mainly due to an energy deviation, $x_0 = D\epsilon$.

The derivative of the total energy loss U in the whole machine is

$$\frac{dU}{d\epsilon} = U_0 \left(J_{\epsilon 0} + \frac{L_w K'' D^3 \rho_0}{2\pi \rho_w} \frac{U_L}{U_0} \epsilon^2 \right) \quad (3)$$

where $J_{\epsilon 0}$ is the longitudinal damping partition for small amplitudes determined by the lattice and a dipole-quadrupole wiggler; U_0 is the total energy loss per revolution for particles with $\epsilon = 0$ and U_L is the (usually dominant) energy loss in the lattice dipole magnets with bending radius ρ_0 alone. Inserting (3) into (1) gives

$$\ddot{\epsilon} - \lambda \Omega_s \left(1 + \frac{b}{J_{\epsilon 0}} \epsilon^2 \right) \dot{\epsilon} + \Omega_s^2 \epsilon = 0 \quad (4)$$

with

$$-\lambda \Omega_s = \frac{\alpha_0 J_{\epsilon 0}}{2} = \frac{\omega_0}{2\pi} \frac{U_0}{E_0} J_{\epsilon 0}; \quad b = \frac{L_w K'' D^3 \rho_0}{2\pi \rho_w} \frac{U_L}{U_0}$$

Here α_0 is the total damping rate of all modes of oscillation, b is the effective strength of the dipole-octupole wiggler which is assumed to be positive so that the longitudinal damping increases with amplitude. The parameter λ gives the relative strength of the longitudinal damping compared to Ω_s ; it is assumed that $|\lambda| \ll 1$. For $J_{\epsilon 0} < 0$ the small amplitudes are antidamped and (4) is the van der Pol equation. It has the approximate solution

$$\begin{aligned} \epsilon(t) &= A(t) \cos(\Omega_s t + \phi) \quad ; \\ A^2(t) &= \frac{A_0^2}{1 + [(A_0/A(t))^2 - 1] \exp(-\lambda \Omega t)} \end{aligned} \quad (5)$$

In this case the amplitude starts with the initial value $A(0)$ at $t = 0$ and approaches asymptotically the so-called limit cycle A_0

$$A(\infty) = A_0 = \sqrt{\frac{4J_{\epsilon 0}}{b}} \quad (6)$$

from above or below depending on whether the initial amplitude is larger or smaller than A_0 . For $J_{\epsilon 0} \geq 0$ solution (5) is still valid but A_0^2 is negative and does not represent a limit cycle.

3. Longitudinal Particle Distribution

Due to the quantum fluctuations of the synchrotron radiation the electrons will not stay on the limit cycle but form a distribution around it. This is calculated by replacing the deterministic equation (4) by a stochastic differential equation

$$\ddot{\epsilon} - \lambda \Omega_s \left(1 + \frac{b}{J_{\epsilon 0}} \epsilon^2 \right) \dot{\epsilon} + \Omega_s^2 \epsilon = \lambda \Omega_s^2 \xi(t) \quad (7)$$

where $\xi(t)$ is a delta-correlated stochastic process with zero mean value. Its magnitude is determined in such a way that for 'normal' conditions $J_{\epsilon 0} = 2$, $b = 0$ the 'normal' energy spread $(\sigma_E/E)_0$ is obtained. Equation (7) has been solved by combining the Bogoliubov-Krylov averaging method⁶ with the modern theory of stochastic processes⁷. The details of this solution are described in a separate paper⁸ and only the results are given here. The method leads to a Fokker-Planck equation for the distribution $W(A, \phi)$, A

and ϕ being amplitude and phase as in (5) but containing also small terms of the order of λ :

$$\begin{aligned} \frac{2}{\lambda \Omega_s} \dot{W}(A, \phi) = & - \frac{\partial}{\partial A} \left[\left(1 - \left(\frac{A}{A_0} \right)^2 - \frac{2}{J_{E0}} \left(\frac{\sigma_E}{E} \right)_0 \frac{1}{A^2} \right) W(A, \phi) \right] \\ & + \frac{\lambda}{4} \frac{\partial}{\partial \phi} \left[\left(1 - 6 \left(\frac{A}{A_0} \right)^2 + \frac{11}{2} \left(\frac{A}{A_0} \right)^4 \right) W(A, \phi) \right] \\ & - \frac{2}{J_{E0}} \left(\frac{\sigma_E}{E} \right)_0 \left(\frac{\partial^2}{\partial A^2} + \frac{1}{A^2} \frac{\partial^2}{\partial \phi^2} \right) W(A, \phi) \end{aligned}$$

Integrating over ϕ gives the distribution $W(A)$ in amplitude alone. Assuming $|\lambda| \ll 1$ and setting $\dot{W}(A) = 0$ gives the stationary distribution

$$W(A) = \frac{4r^2 A}{A_0^2 g(r)} \exp \left[2r^2 \left(\frac{A}{A_0} \right)^2 \left(1 - \frac{1}{2} \left(\frac{A}{A_0} \right)^2 \right) \right] \quad (8)$$

with

$$\int_0^\infty W(A) dA = 1; \quad r = \frac{-J_{E0} E_0}{2\sqrt{b} \sigma_E}; \quad g(r) = \sqrt{\pi} r \operatorname{erfc}(-r) e^{r^2} \quad (9)$$

Expression (8) gives the distribution with respect to the amplitude A and is related to the phase space density $F(A)$ by

$$W(A) = 2\pi A F(A).$$

If $J_{E0} > 0$ the density $F(A)$ has a maximum in the centre $A = 0$; for $J_{E0} < 0$ it has a minimum in the centre and a maximum on the limit cycle $A = A_0$ (Fig. 1). The mean square amplitude is

$$\langle A^2 \rangle = A_0^2 (1 + 1/g(r)). \quad (10)$$

Of special interest is the projection of the distribution $F(A)$ on the longitudinal axis which gives the line density $\Lambda(s)$ with respect to the longitudinal coordinate s .

$$s = \frac{R\alpha}{Q_s} \frac{\dot{\epsilon}}{\Omega_s}, \quad Q_s = \frac{\Omega_s}{\omega_0} \quad (11)$$

with R = average radius of the machine, α = momentum compaction factor. Defining

$$q = \frac{r}{2} \left[\left(\frac{Q_s s}{R\alpha A_0} \right)^2 - 1 \right]$$

we get⁹

$$\Lambda(s) = \frac{Q_s}{R\alpha} \int_{-\infty}^{+\infty} F(A) d\epsilon = \frac{\sqrt{-qrJ_{E0}}}{\pi \Omega_s g(r)} e^{r^2} e^{-2q^2} K_{\frac{1}{2}}(2q^2) \quad (12)$$

Caution: if $q < 0$ the analytic continuation for $K_{\frac{1}{2}}$ has to be used

$$K_{\frac{1}{2}}(2q^2 e^{2\pi i}) = -i [K_{\frac{1}{2}}(2q^2) + \pi \sqrt{2} I_{\frac{1}{2}}(2q^2)]$$

$K_{\frac{1}{2}}$ and $I_{\frac{1}{2}}$ are modified Bessel functions of order $\frac{1}{2}$ and σ_s^2 is the rms bunch length for 'normal' conditions ($b = 0$, $J_{E0} = 2$). The line density $\Lambda(s)$ has a maximum in the centre if $r < 0.541$ and a minimum if $r > 0.541$. The mean square bunch length is obtained from (10) and (11)

$$\langle s^2 \rangle = \frac{1}{2} \left(\frac{R\alpha A_0}{Q_s} \right)^2 \left(1 + 1/g(r) \right)$$

The phase space density and the line density are shown in Fig. 1 for $r = 0.541$.

In a realistic machine the energy acceptance is limited and particles with amplitudes exceeding a certain value A_m will be lost. This leads to a quantum lifetime⁸ which can be estimated from

$$\frac{1}{\tau_q} \approx \frac{1}{n} \frac{dn}{dt} = W(A_m) \left| \frac{dA}{dt} \right| = W(A_m) |\lambda| \Omega_s (1 - (A_m/A_0)^2)$$

Expressing A_m in units of the 'normal' energy spread $A_m = m(\sigma_E/E)_0$ and introducing $\xi = m^2 J_{E0}/4$ an expression is obtained which relates the quantum lifetime τ_q to the total damping rate α_0

$$\frac{\tau_q \alpha_0}{m^2} = \frac{g(r)}{2\xi^2 (1 + \xi/2r^2)} \exp [\xi(1 + \xi/4r^2)] \quad (13)$$

and which permits the choice of wiggler parameters in order that the lifetime is acceptable.

The expressions of this chapter are presented in a convenient form for the case $J_{E0} < 0$. For the 'normal' case ($J_{E0} = 2$, $b = 0$) one gets $r = -\infty$, $A_0^2 = -\infty$, $g(r) = -1$ and all expressions take the normal form for Gaussian bunches.

4. Transverse Particle Distribution

The damping rates of all three modes of oscillation fulfil Robinson's sum rule², $\alpha_x + \alpha_y + \alpha_z = \alpha_0 = \text{constant}$ or $J_E + J_x + J_y = 4$. Since the vertical motion is not affected by the wiggler, $J_E + J_x = 3$. If the betatron amplitude in the wiggler is small the longitudinal distribution is not affected by the transverse amplitude but the horizontal distribution is influenced by the longitudinal one through the damping rates. For finite betatron amplitudes in the wiggler the horizontal excursion x_0 in (2) is given by

$$x_0 = D\epsilon + x = D(\epsilon + \delta).$$

The longitudinal and transverse equations become

$$\ddot{\epsilon} + \frac{\alpha_0}{2} (J_{E0} + b(\epsilon^2 + \delta^2)) \dot{\epsilon} + \Omega_s^2 \epsilon = 0 \quad (14)$$

$$\ddot{\delta} + \frac{\alpha_0}{2} (J_{x0} - b(\epsilon^2 + \delta^2)) \dot{\delta} + \Omega_x^2 \delta = 0$$

where $J_{x0} = 3 - J_{E0}$ is the transverse damping for small energy and betatron amplitudes. To derive (14) it was assumed that $\epsilon \ll 1$, $\delta \ll 1$ and that the damping is weak so that odd powers of ϵ and δ average to zero over one oscillation. Furthermore the frequencies of the two modes should be different so that they have no common sidebands and terms with $\epsilon\delta$ average to zero. The two equations (14) are coupled with each other via the damping term. Writing the two modes of oscillation as

$$\epsilon(t) = A \cos(\Omega_s t + \phi), \quad \delta(t) = \Delta \cos(\Omega_x t + \theta)$$

one finds that for amplitudes A and Δ exceeding the unstable limit cycle

$$2A^2 + \Delta^2 \approx \frac{4J_{x0}}{b} \quad (15)$$

the transverse oscillation becomes antidamped. Under these conditions only a quasi-stationary distribution can be obtained. The parameters of the wiggler have to be chosen so that the unstable limit cycle (15) is far out where the particle density is extremely small.

The methods mentioned in the last chapter have been used to solve the coupled stochastic equations and gave a four-dimensional Fokker-Planck equation. The details of this derivation are described in a separate paper⁸.

To obtain compact analytical expressions some approximations had to be made which are only valid for amplitudes A and Δ well inside the unstable limit cycle (15). The transverse distribution becomes a Gaussian with a correction

$$V(x) \approx \frac{J_{x_0} \exp \left[-\frac{x^2 J_{x_0}}{2\sigma_{x_0}^2} \right] \left(1 + \frac{x^2 b}{4\sigma_{x_0}^2} \left(\langle A^2 \rangle + \frac{x^2}{4D^2} \right) \right)}{2\sigma_{x_0}^2 \left(1 + \frac{b}{4J_{x_0}} \left(\langle A^2 \rangle + \frac{\sigma_{x_0}^2}{4D^2 J_{x_0}} \right) \right)} \quad (16)$$

where σ_{x_0} is the rms beam size in the wiggler due to betatron oscillation alone under 'normal' conditions ($J_{x_0} = 1$, $b = 0$). The mean square betatron amplitude in the wiggler is

$$\langle \Delta^2 \rangle = \frac{2\langle x^2 \rangle}{D^2} \approx \frac{2\sigma_{x_0}^2}{D^2 J_{x_0}} \left(1 + \frac{b}{2J_{x_0}} \left(\langle A^2 \rangle + \frac{2\sigma_{x_0}^2}{D^2 J_{x_0}} \right) \right) \quad (17)$$

The longitudinal distribution is obtained within this approximation by making the following substitutions in the equations of the last chapter

$$r \rightarrow r \left(1 - \frac{2\langle \Delta^2 \rangle}{A_0^2} \right) ; \quad A_0^2 \rightarrow A_0^2 \left(1 - \frac{2\langle \Delta^2 \rangle}{A_0^2} \right) \quad (18)$$

Since $\langle \Delta^2 \rangle$ and $\langle A^2 \rangle$ are both contained in each distribution some iteration might be necessary to determine them. In any case equations (16) to (18) are only approximations, useful for estimating the distribution and optimizing the wiggler parameters. A more accurate distribution could be obtained from numerical calculations.

It might be feasible to use a wiggler having a field shape which again gives positive damping for all modes at very large amplitudes and avoids the unstable limit cycle.

5. Computer Simulation

The longitudinal particle distribution can easily be obtained by 'tracking' the motion of a large number of particles in synchrotron phase space¹⁰. The dipole-octupole wigglers cause the energy loss per turn to vary as

$$\Delta U = a_0 + a_1 \frac{\partial E}{E} + a_3 \left(\frac{\partial E}{E} \right)^3$$

In order to investigate the change in the transverse distributions the 'tracking' must be performed in six-dimensional phase space. To this end a wiggler insertion was built into an already existing six-dimensional 'tracking' program¹¹. Horizontally the transition through the wiggler and an RF station is given by

$$\begin{aligned} x_1 &= x_0 + \frac{D U_w}{E} \\ x_1' &= x_0' - \frac{e \Delta V_{RF} x_0'}{E} \end{aligned}$$

where U_w is the loss in the wiggler and $e\Delta V_{RF}$ is the energy gain in the RF cavity. The quantum excitations which occur in the machine arcs are unchanged. This simple technique causes the damping partition numbers to be related through the wiggler losses and therefore needs no assumptions about the sum rule for the partition numbers.

The results obtained from the computer simulation agree very well with the distributions predicted by the analytical approach.

6. Example and Conclusion

As an illustrating example a dipole octupole wiggler is considered for LEP operating at injection energy of 22 GeV with a 'normal' rms energy spread $(\sigma_E/E)_0 = 1.24 \times 10^{-3}$. To obtain a smooth distribution $r = 0.541$ is chosen. Neglecting transverse effects and assuming an available energy aperture of

$\pm 10(\sigma_E/E)_0$ and a desired quantum lifetime $\tau_q = 10$ h one gets for the limit cycle $A_0 = 4.3 \times 10^{-3}$, the small amplitude damping partition $J_{E0} = -0.2$ and an effective wiggler strength $b = 4.22 \times 10^4$.

So far no design for the wiggler magnet exists; however as a rough estimate an octupole with $\partial^3 B / \partial x^3 \approx 3000$ T/m³ is assumed which gives a maximum field of 0.5 T in a circular aperture of 0.1 m radius. A dipole field of 0.5 T is superimposed. Taking furthermore a dispersion of $D = 3$ m a total magnetic length of $L_w = 25$ m will be necessary.

This wiggler is clearly a big object and only feasible in conjunction with a large dispersion. However it produces a more favourable longitudinal particle distribution with an rms energy spread or bunch length being three times larger than the 'normal' values. This might help the beam stability at injection. The transverse effects may reduce the required wiggler strength somewhat. We hope that an optimized magnet design will help also. To avoid excitation of non-linear resonances by the octupole field, the wiggler has to contain many sections of alternating polarity within one betatron wavelength.

Acknowledgement: We would like to thank B. Zotter for his help in solving some mathematical problems.

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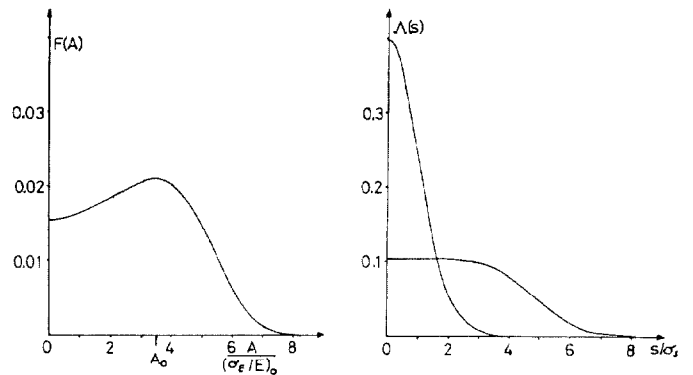


Fig. 1. Phase space density $F(A)$ and line density $\Lambda(s)$ against the amplitude A and the longitudinal coordinate s (both in units of 'normal' rms values) for a wiggler with $r = 0.541$ as described in chapter 6. For comparison a 'normal' Gaussian line density is shown.