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THEORETICAL TREATMENT OF TRANSVERSE FEEDBACK SYSTEMS WITH MEMORY*

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1. INTRODUCTION

The differential equation of the dipole moment of coherent oscillations in the presence of a feedback system is derived. The analysis, which starts in the time domain, is extended to the frequency domain; this allows a straightforward derivation of the damping rate for both coasting and bunched beams. The damping rate is expressed in terms of the transfer function of the feedback system and in a general form which takes into account the β-function and betatron phase modulation along the beam trajectory, the effect of memory arising from the finiteness of the system bandwidth, the effect of the time delay and of the betatron phase advance between detector and kicker. Some examples of the dependence of the damping rate on the feedback parameters are given.

2. DESCRIPTION OF THE SYSTEM AND DEFINITION OF THE SYMBOLS

The schematic layout of the system is shown in Fig. 1. The detector (D) monitors coherent transverse (horizontal or vertical) oscillations of the beam. A voltage signal proportional to the beam displacement in D is transferred, with an appropriate delay, to the kicker (K) which deflects the beam. The symbols used in the analysis are the following:

- t = time
- $\omega_o/2\pi$ = revolution frequency
- R = ring mean radius
- s = distance measured along the beam closed orbit
- θ = azimuthal angle around the ring (increases by 2π in one revolution).
- ν_o = betatron tune in absence of damping
- ν = betatron tune with damping
- β = β- function
- ϕ = betatron phase = $1/\nu_o \int_0^s [ds/\beta(s)]$
- ϕ_i = betatron phase of the i-th particle
- ψ_i = initial azimuthal angle of the i-th particle = $\theta_i(t) - \omega_o t$
- x_i = amplitude of oscillation of the i-th particle
- D = dipole moment.

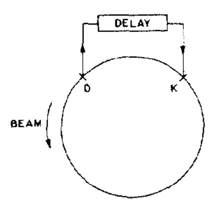

Fig. 1. Schematic layout of a feedback system.
D = detector;
K = kicker.

3. THEORETICAL TREATMENT FOR COASTING BEAMS

Let's express the displacement of the i-th particle in the form

$$x_i(t) = \sqrt{\beta[\phi_i(t)]}\ \xi_i e^{-j\nu[\phi_i(t)-\psi_i]} \qquad 1)$$

where the normalized amplitude (in the Courant-Snyder sense[1]) ξ_i can be complex, and is independent of time.

The dipole moment $D(\theta,t)$ is given by

$$D(\theta,t) = \Sigma_i x_i(t)\delta_p(\theta-\omega_o t-\psi_i) =$$

$$= \Sigma_i \sqrt{\beta[\phi_i(t)]}\ \delta_p(\theta-\omega_o t-\psi_i)\xi_i e^{-j\nu[\phi_i(t)-\psi_i]} \qquad 2)$$

where δ_p is the periodic δ-function, and the summation over i extends to all the particles in the beam.

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Since, when $\theta=\omega_o t+\psi_i$, $\phi_i(t)=\phi(\theta)$, Eq. 2) can be written

$$D(\theta,t) = \sqrt{\beta(\phi)}\ e^{-j\nu\phi}\Sigma_i \xi_i e^{j\nu\psi_i}\delta_p(\theta-\omega_o t-\psi_i) \qquad 2a)$$

If the longitudinal density is constant along the ring, then we have, for the n-th harmonic of the collective motion (in the variable $\psi = \theta - \omega_o t$)

$$\lim_{\Delta\to 0} \frac{1}{\Delta} \Sigma_{\psi<\psi_i<\psi+\Delta} \xi_i = \frac{N}{2\pi}\bar{\xi}\ e^{-jn\psi} \qquad 3)$$

where $\bar{\xi}$ is constant and independent of ψ, and N is the total number of particles around the ring. With the assumption implied in Eq. 3), Eq. 2a) becomes

$$D_n(\theta,t) = \frac{N}{2\pi}\bar{\xi}\sqrt{\beta(\phi)}\ e^{-j\nu\phi}e^{-j(n-\nu)(\theta-\omega_o t)} =$$

$$= \frac{N}{2\pi}\bar{\xi}\sqrt{\beta(\phi)}\ e^{-j\nu(\phi-\theta)}e^{-jn\theta+j(n-\nu)\omega_o t} \quad . \qquad 4)$$

Equation 4) recalls the familiar expression of the coherent motion in a "smooth" machine, modified by the term $\sqrt{\beta(\phi)}e^{-j\nu(\phi-\theta)}$, which takes into account the modulation due to the strong focusing. Equation 4) can be written in terms of the variables ϕ,ψ

$$D_n(\phi,\psi) = \frac{N}{2\pi}\bar{\xi}\sqrt{\beta(\phi)}\ e^{-j\nu(\phi-\psi)}e^{-jn\psi} \quad . \qquad 5)$$

From the above equation we see that the function $E_n(\phi,\psi)=D_n(\phi,\psi)/\sqrt{\beta(\phi)}$ satisfies the differential equation

$$\frac{\partial^2 E_n}{\partial\phi^2} + \nu^2 E_n = 0 \qquad 6)$$

In the absence of a feedback system or any other transverse impedance, the betatron frequency ν in Eq. 6) is equal to the unperturbed value ν_o.

In the presence of a feedback system, let the force field in the kicker be

$$a(s,t) = A(t)\delta(s-s_K) = \frac{1}{\nu_o\beta_K} A(t)\delta(\phi-\phi_K) \qquad 7)$$

where s_K and ϕ_K denote the position of the kicker, β_K the β-function value at the kicker location. In this case, Eq. 6) can be written

$$\frac{\partial^2 E_n}{\partial\phi^2} + \nu_o^2\ E_n(\phi,\psi) = \frac{N}{2\pi}\ \nu_o\beta_K^{1/2}\ A(t)\delta_p(\phi - \phi_K) \qquad 8)$$

If F(t) is the Green's function of the chain detector-kicker,

$$A(t) = \int_{-\infty}^{\infty} F(t-t')\ D_n(\theta_D,t')\ dt' \qquad 9)$$

where θ_D is the azimuthal position of the detector. Denoting by $\tilde{F}(\omega)$ the Fourier transform of F(t), and substituting Eq. 4) into Eq. 9), we obtain

$$A(t) = N\ \bar{\xi}\sqrt{\beta_D}\ e^{-j\nu_o(\phi_D-\theta_D)}\ e^{-jn\theta_D}\ \cdot$$

$$\cdot\ \tilde{F}[(n-\nu_o)\omega_o]e^{j(n-\nu_o)\omega_o t} \quad . \qquad 10)$$

We may now substitute Eq. 10) into Eq. 8), convert the equation in the (ϕ,ψ) variables and average over one revolution. Since, on average,

$$e^{j(n-\nu_o)\omega_o t}\delta_p(\phi-\phi_K) \simeq e^{-j(n-\nu_o)\psi}e^{-j\nu_o\phi}e^{-j\nu_o(\theta_K-\phi_K)}\cdot$$

$$\cdot\ e^{jn\theta_K}\frac{1}{2\pi} \quad ,$$

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Eq. 8) becomes

$$\frac{\partial^2 E_n}{\partial\phi_2}+\nu_o^2 E_n=\frac{\nu_o N}{2\pi}\sqrt{\beta_D\beta_K}\tilde{F}[(n-\nu_o)\omega_o]e^{-j\nu_o[(\phi_D-\theta_D)-(\phi_K-\theta_K)]}\cdot e^{-jn(\theta_D-\theta_K)}E_n \qquad 8)$$

The perturbed frequency ν is thus given by

$$\nu_o^2-\nu^2=\frac{\nu_o N}{2\pi}\sqrt{\beta_D\beta_K}\,\tilde{F}[(n-\nu_o)\omega_o]e^{-j\nu_o[(\phi_D-\theta_D)-(\phi_K-\theta_K)]}\cdot e^{-jn(\theta_D-\theta_K)} \qquad 9)$$

The imaginary part of ν gives the damping rate.

4. DISCUSSION ON THE EQUIVALENT IMPEDANCE OF THE FEEDBACK SYSTEM

Equation 9) states that the damping rate depends on the Fourier transform of the Green function of the chain detector-kicker. In analogy with the definition of impedance which is used in describing collective phenomena, this function represents the impedance of the feedback system. This concept of impedance of a feedback system is, of course, well known. However, there are certain aspects which relate it to the temporal behavior of the response of the system which are worth discussing.

Let $F(t)$ be the Green's function of the system, with $F(t) = 0$ for $t<\tau$, $\tau > 0$, and let $G(t) \equiv F(t+\tau)$. From the definition of Fourier transform

$$\tilde{F}(\omega) = \int_\tau^\infty F(t)e^{-j\omega t}dt \quad ,$$

$$\tilde{G}(\omega) = e^{-j\omega\tau}\tilde{F}(\omega) \quad .$$

As an example, let's consider the system depicted in Fig. 2. The kicker consists of a magnetic deflector; thus $\tilde{G}(\omega)$ is the frequency response of the current in the kicker to an input voltage at its input (2, 2'). $\tilde{F}(\omega)$ is the current frequency response to a voltage signal at the detector (1,1'). Let's assume that the transfer function detector output (1,1')÷ kicker input (2,2') consists of a pure delay τ, and let's express this delay as

$$\tau = T_o - \frac{\theta_D-\theta_K}{\omega_o} - \varepsilon \qquad 10)$$

where T_o = revolution period. Thus, $\varepsilon = 0$ represents the case in which there is coincidence between the beam reaching the kicker position and the signal coming from the detector and induced by the beam. Replacing 10) into 9), the betatron tune can be expressed in a useful form showing the explicit dependence on ε:

$$\nu_o^2-\nu^2=\frac{\nu_o N}{2\pi}\sqrt{\beta_D\beta_K}\;e^{j(\mu-\mu_D+\mu_K)}e^{j\varepsilon(n-\nu_o)\omega_o}\tilde{G}[(n-\nu_o)\omega_o] \qquad 11)$$

where $\mu-\mu_D+\mu_K$ is the betatron phase advance from the detector to the kicker.

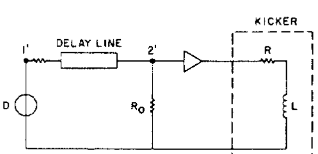

Fig. 2. Circuit diagram of a feedback system consisting of the detector output (D), a matched delay line, an ideal amplifier and a magnetic kicker schematized by an inductance in series with a resistance.

If the Green's function of the kicker is such that $G(t)=0$ when $t=0$, then, from the definition of Fourier transform,

$$\int_{-\infty}^{\infty}\tilde{G}(\omega)d\omega = 0 \qquad 12)$$

Thus some of the coasting beam modes will always be antidamped. Increasing the rise time of the kicker response to a δ-function pushes the antidamping terms towards higher frequencies.

5. INTERPRETATION OF THE PHASE ADVANCE DETECTOR KICKER IN THE FREQUENCY DOMAIN

Since $G(\omega)$ has the property, deriving from causality and the reality of $G(t)$

$$G^*(-\omega) = G(\omega)$$

it follows, if $G = U + jV$,

$$U(\omega) = U(-\omega)$$
$$V(\omega) = -V(-\omega)$$

condition which is common to all beam impedances. We see from Eq. 11) that a phase advance between detector and kicker which is an odd multiple of 90°, and perfect coincidence (ε=0), make the U part responsible for damping, thus allowing damping of both positive and negative frequencies. Any departure from 90° tends to decrease the effectiveness of the damping of the fast wave, and it may even lead to antidamping.

6. DAMPING OF SYMMETRIC COUPLED BUNCH MODES

A derivation similar to the one described in Section 3 has been applied to a bunched beam describing coherent oscillations. Each bunch is treated as a δ-function of charge, and the betatron phase shift between consecutive bunches is $2\pi s/B$, where B is the number of bunches and s the mode number ($s=0,1,2\ldots, B-1$).

In this case, the perturbed betatron frequency ν of the s-th mode is given by the equation

$$\nu_o^2-\nu^2=\frac{\nu_o N}{2\pi}\sqrt{\beta_D\beta_K}e^{j(\mu-\mu_D+\mu_K)}\sum_{\ell=-\infty}^{\infty}e^{j(B\ell+s-\nu_o)\omega_o\varepsilon}\cdot\tilde{G}[(B\ell+s-\nu_o)\omega_o] \qquad 13)$$

Equation 13) can be converted in the time domain, yielding

$$\nu_o^2-\nu^2=\frac{\nu_o N}{2\pi B\omega_o}\sqrt{\beta_D\beta_K}\;e^{j(\mu-\mu_D+\mu_K)}\cdot\sum_{\ell\geq-\frac{\varepsilon B}{T}}G(\varepsilon+\frac{\ell T}{B})e^{-j(s-\nu_o)(2\pi\ell/B)} \qquad 14)$$

The choice between Eqs. 13) and 14) depends on the characteristics of the kicker response: the faster the decay time response to a δ-function drive, the smaller the number of significant terms in the summation of Eq. 14) compared to Eq. 13).

From the discussion of the previous section, we see that if the phase advance between detector and kicker is an odd multiple of 90° and ε=0, all the coupled bunch modes can be damped. Any departure from the above conditions may lead to some modes being antidamped.

7. DEPENDENCE OF DAMPING RATE ON SOME FEEDBACK PARAMETERS

We have applied the results of Eqs. 11), 13), 14), to the study of the type of feedback system depicted in Fig. 2. This is an ideal system where the delay line introduces a pure real delay τ and the kicker is approximated by a resistance (R) in series with an inductance (L). The current in the kicker (thus the magnetic field responsible for the deflection) is related to an input voltage at the detector (proportional to the amplitude of the coherent oscillations) via the frequency response

$$\tilde{F}(\omega) = K\frac{e^{-j\omega\tau}}{1+j\omega\tau_o} \qquad 15)$$

where K is a constant which includes the gain of the system, τ is the delay, τ_o=L/R is the time constant of the kicker. The Green's function of the kicker is $G(t)=e^{-t/\tau_o}$; thus, for such an ideal system, $G(0)=1$, and, following the argument of the previous Section, it is possible to damp all coasting beam modes.

Figures 3a), 3b) show the damping rate (in arbitrary units) as a function of the coasting beam mode number. We have assumed (ISABELLE case), ν_o=22.6; the time constant of the kicker, τ_0, is 1/57 of the revolution period. The curves of Fig. 3a) refer to three different values of the delay: $\varepsilon=0$ (coincidence), $\varepsilon=0.05\tau_0$ (kicker signal anticipates the beam), $\varepsilon = -0.05\tau_0$. It is interesting to note that the curve falls off less rapidly when the detector signal slightly anticipates the beam ($\varepsilon>0$). This, however, has the consequence that, for larger mode numbers $[|n|>\sim(\pi/\omega_0\varepsilon)\underline{+}\nu_0]$, the damping rate changes sign, thus antidamping. In Fig. 3b), we show the effect of a phase advance detector÷kicker different from an odd multiple of 90˘: obviously a large deviation from this optimum condition introduces antidamping. Figures 4a), 4b) show the effect of the kicker time constant and phase advance detector÷kicker on a bunched beam. The calculations were carried out for 57 bunches (∿ the number of bunches in ISABELLE at injection) and the kicker time constant is expressed as a fraction of the time separation between consecutive bunches. The dependence of the damping time on the delay is shown in Fig. 4c): for $\varepsilon>0$ (driving signal at the kicker precedes the beam), the only effect is a damping rate reduction by the term $e^{-\varepsilon/\tau_0}$ (provided ε is less than the bunch separation); for $\varepsilon<0$ (beam precedes the signal) the effect is more serious, and some of the modes are actually antidamped.

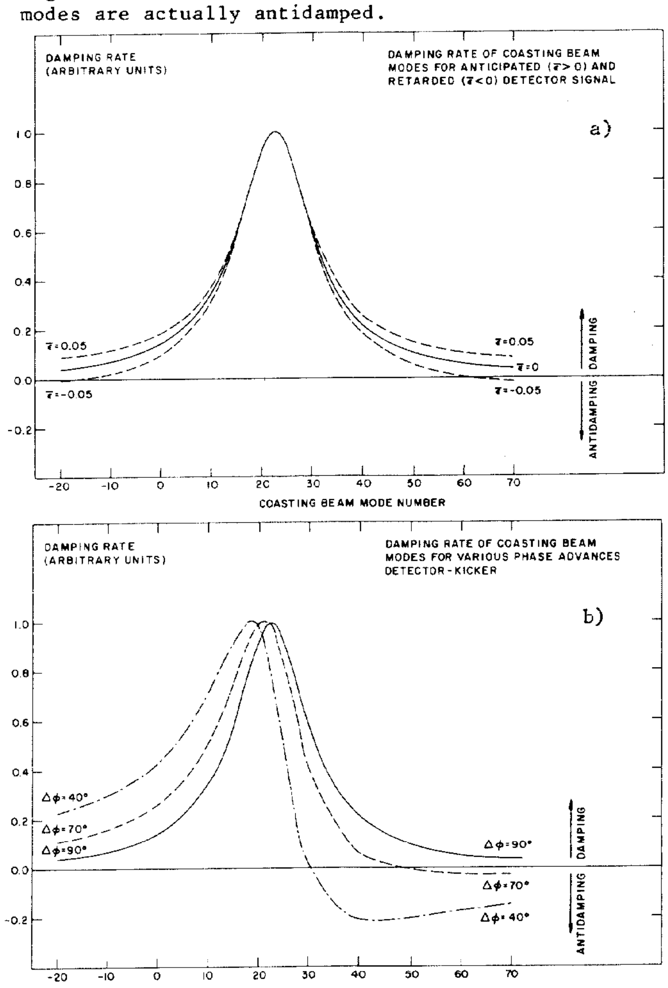

Fig. 3. Damping rate of coasting beam modes for various delays and betatron phase advances, $\Delta\phi$, between detector and kicker. The delay $\bar{\varepsilon}$ is expressed as a fraction of the kicker time constant, $\bar{\varepsilon}=\varepsilon/\tau_0$. For each curve, the damping rate is normalized to the maximum value.
a) $\Delta\phi = 90˘$; $\tau_0 = 1/57$ revolution frequency
b) $\varepsilon = 0$; $\tau_0 = 1/57$ revolution frequency.

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1. E.D. Courant and H.S. Snyder, "Theory of Alternating Gradient Synchrotron", Ann. Phys. 3, 1 (1958).

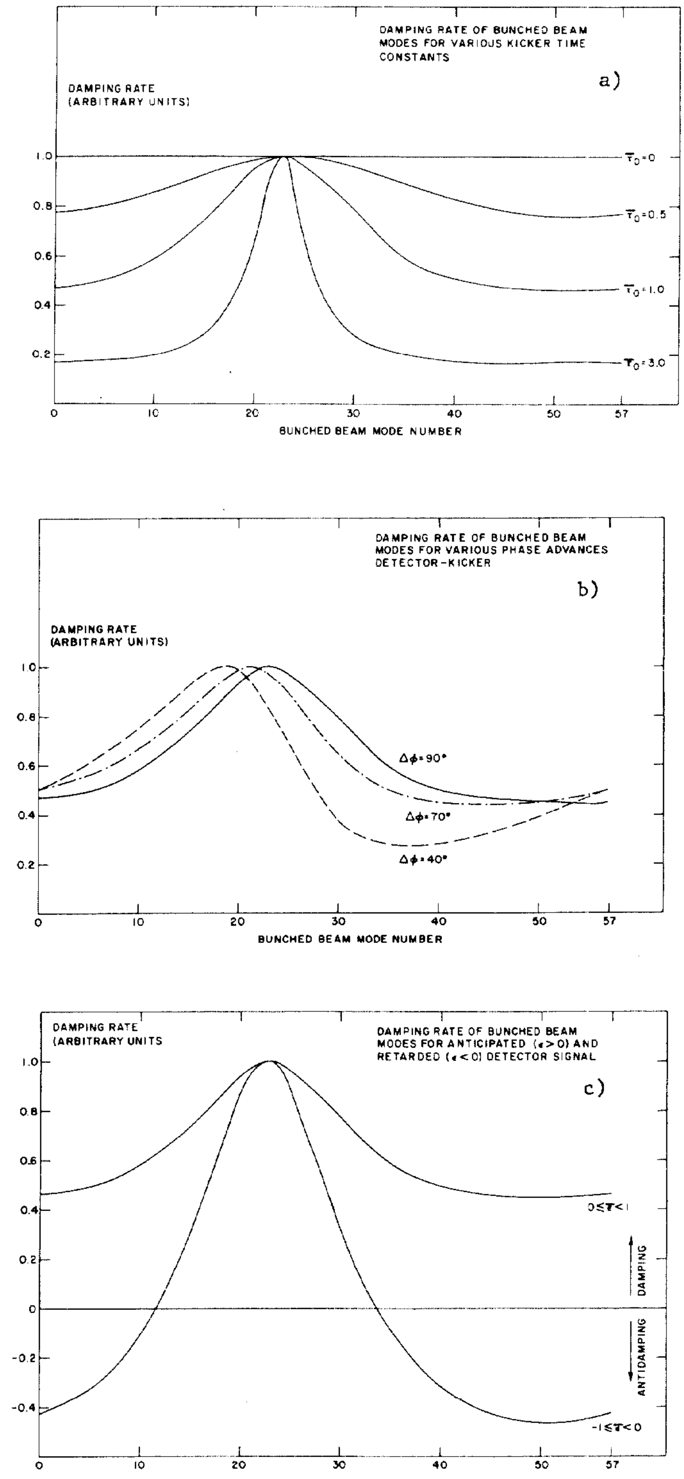

Fig. 4. Damping rate of bunched beam modes for various:
a) Kicker time constant, $\bar{\tau}_0$, expressed as a fraction of the bunch separation: $\bar{\tau}_0 = 57(\tau_0\omega_0/2\pi)$. $\Delta\phi=90^o$; $\varepsilon=0$.
b) Betatron phase advance detector÷kicker, $\Delta\phi$, $\bar{\tau}_0 = 1$; $\varepsilon = 0$.
c) Delay $\bar{\varepsilon}$, expressed as a fraction of the bunch separation: $\bar{\varepsilon} = 57(\varepsilon\omega_0/2\pi)$. $\Delta\phi = 90˘$; $\bar{\tau}_0 = 1$.