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AIR CORE BETATRONS REEXAMINED

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Summary

Theoretical studies on the possibility of increasing both current and energy of the air core betatron indicate (1) the principal current limit is set by the betatron radius, which should be small for higher currents (2) the principal energy limit is also set by the radius, with a large radius preferred, but also by the energy storage and the time profile of the voltage applied to the primary coils, together with the mutual inductance per turn, M', of the primary and secondary coils. The "Helmoltz-coil" configuration appears to be superior in maximizing M'.

Current

Equations

Using the Kapchinskii-Vladimirskii distribution,[1] the radial (x) and axial (y) equations of a relativistic charged particle can be written, for a weak-focused lattice, as:

$$\frac{d}{ds}(\eta x') + \frac{(1-n)\eta}{\rho^2} x - \frac{4Nr_e x}{\eta^2(X+Y)X} = 0 \quad (1a)$$

$$\frac{d}{ds}(\eta y') + \frac{n\eta}{\rho^2} y \frac{4Nr_e y}{\eta^2(X+Y)Y} = 0 \quad (1b)$$

in which n = field index

$\eta = \beta(1-\beta^2)^{-\frac{1}{2}}$ (momentum)

N = no. of electrons per unit length

r_e = classical electron radius

ρ = betatron radius

X = radial envelope

Y = axial envelope

s = distance traveled

With the aid of the emittance-momentum relationship:

$$\varepsilon\eta = \text{constant} \quad (2)$$

(emittance shrinkage caused by sychrotron radiation is neglected) and the relations between the Courant-Synder parameters $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ [2]

$$\bar{\gamma}\bar{\beta} - \bar{\alpha}^2 = 1 \quad (3)$$

$$\bar{\alpha} = -\frac{\eta}{2}\frac{d}{ds}\left(\frac{\bar{\beta}}{\eta}\right) \quad (4)$$

(where Eq. (4) has been modified from its more familiar form to include effects of acceleration) the equations for the envelopes X and Y have been obtained. These are:

$$\frac{d}{ds}\left(\frac{X'}{\varepsilon_x}\right) + \frac{(1-n)X}{\rho^2\varepsilon_x} - \frac{4Nr_e}{\eta^3\varepsilon_x(X+Y)} - \frac{\varepsilon_x}{X^3} = 0 \quad (5a)$$

$$\frac{d}{ds}\left(\frac{Y'}{\varepsilon_y}\right) + \frac{nY}{\rho^2\varepsilon_y} - \frac{4Nr_e}{\eta^3\varepsilon_y(X+Y)} - \frac{\varepsilon_y}{Y^3} = 0 \quad (5b)$$

Results

The coupled Eqs. (5) have been programmed for solution on the AFWL Cray computer, together with Eq. (2), and an assumed time profile for the accelerating voltage proportional to $(1 - \cos\Omega t)$ where π/Ω is the accelerating time. The ρ value used was always 100 cm, with an initial η of 20 and a final η of 200. Initial emittances ε_x and ε_y were usually 1 cm - mrad, with various iterations on shape and orientation. Field indices, n, were varied from 0.3 to 0.5. At this last value, Eqs. (5) are identical: it was found that the expected coupling resonance could become severe at very low currents but beam behavior became normal for I > 1kA, especially if injected at the "stable envelope" sizes, (X_o, Y_o), obtained by setting the derivatives in Eqs. (5) to zero. "Stable size" injection appears to be the most critical variable for obtaining a well-behaved beam. In Figs. I and II are shown, respectively, the X and Y profiles of a 100 kA electron beam injected at stable X_o and Y_o, while in Figs. III and IV are presented the X and Y profiles of a 30 kA electron beam which was injected at an arbitrary $X_o = Y_o = 1$ cm. All other parameters are identical. Even lower currents exhibit pathological behavior when injected at X_o and Y_o very different from stable values. Similarly, tilt of the input emittance ellipses at injection (X_o', Y_o') away from zero led to serious oscillations, even when injected at stable size, as seen in Fig. V, where the X profile of a 10 kA beam, injected at $X_o' = Y_o' = -10$ mrad, is displayed.

Comment

The maximum permissible current will vary approximately as ρ^{-2}, or at best as ρ^{-1} if vacuum chamber dimensions are scaled with ρ. Thus a small ρ is desireable. The application of strong focusing, so n and (1-n) are replaced by much larger numbers, might be an attractive option if it were not for the additional stability and design problems introduced, especially in an ironless lattice. Additionally, it is noted that the assumed energy gains at the above designs (~ 90 kv/turn) neglect sychrotron radiation and are slightly unrealistic in themselves, as will be shown in the next section.

Energy

Geometry and Inductances

In the schematic air-core betatron shown in Fig. VI, the N_1 primary coils must be arranged symmetrically around the N_2 turns of the secondary (beam). The beam radius, ρ, must be slightly smaller than the primary radius, a, so that the primary magnetic induction will not only accelerate but also steer and focus it. (In a workable design, small passive secondary coils and/or thin ferromagnetic shields would be required to shape the field, and would lead to some variation in the following estimates.)

The coils are shown in a "Helmholtz" geometry. Including the mutual inductance between the two halves, the primary self-inductance per N_1^2 can be shown to be:

$$L_1' = 1/2\,\mu_o a\,[\ln(\frac{8a}{r}) - 0.865] \quad (6)$$

while the self-inductance of the beam per N_2^2 can be calculated by integrating the magnetic flux from the Lienard-Wiechert field and identifying this with (current x inductance). The result is:

$$L_2' = \mu_o\rho f\,[2\ln(\frac{2\rho}{R}) + 1/4] \quad (7)$$

for a completely relativistic beam. Here f represents the fraction of the $2\pi\rho$ circumference occupied by the beam: f becomes unity after some hundreds of turns. R is some average of X and Y and also varies, so L_2' increases in time. The choice of the Helmholtz geometry lead to a nearly constant value of the mutual inductance per N_1N_2 for large ρ changes.

$$M' \simeq \frac{3}{2}\mu_o a \qquad 0.8 < (\rho/a) < 1.3 \quad (8)$$

From Eqs. (6, 7, 8) one can calculate the coupling constants:

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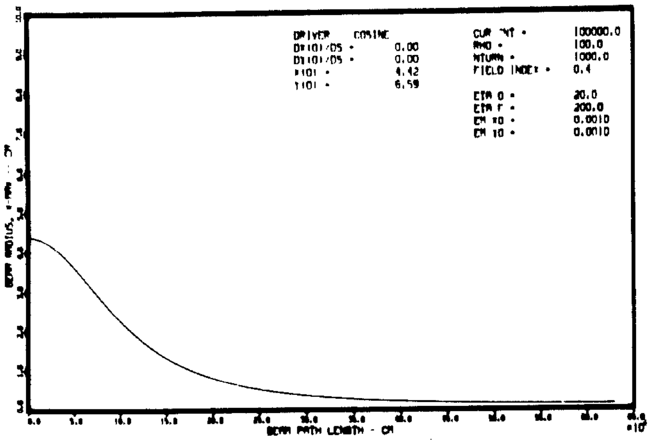

Fig. I. X-Profile for 100 kA at Stable Size

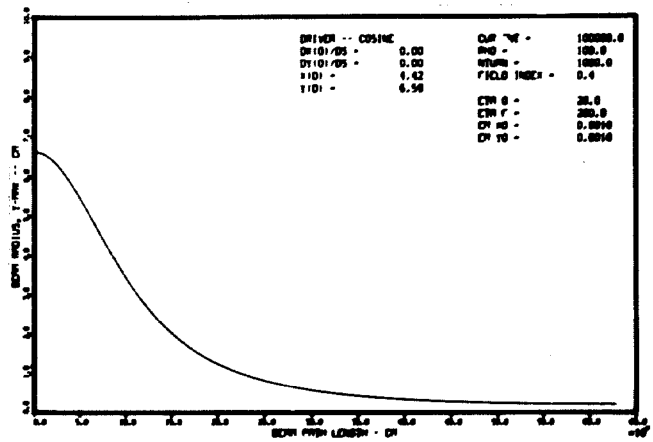

Fig. II. Y-Profile for 100 kA at Stable Size

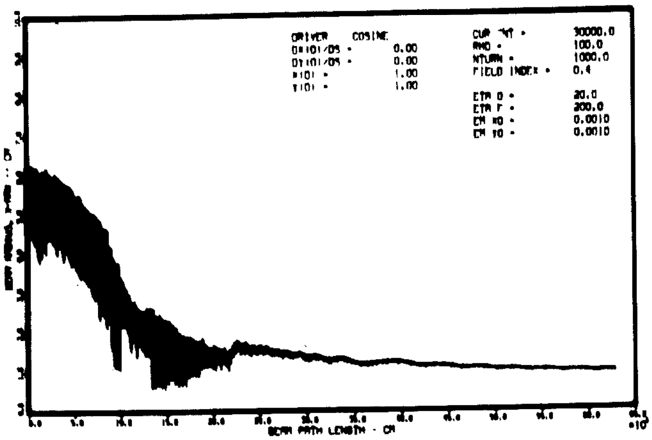

Fig. III. X-Profile for 30 kA at X_0 = 1 cm

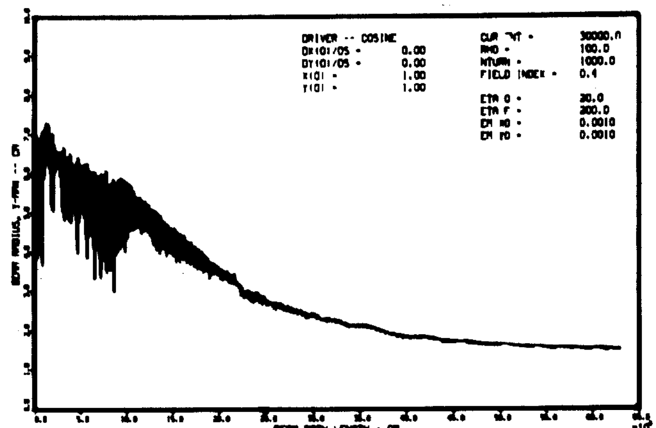

Fig. IV. Y-Profile for 30 kA at Y_0 = 1 cm

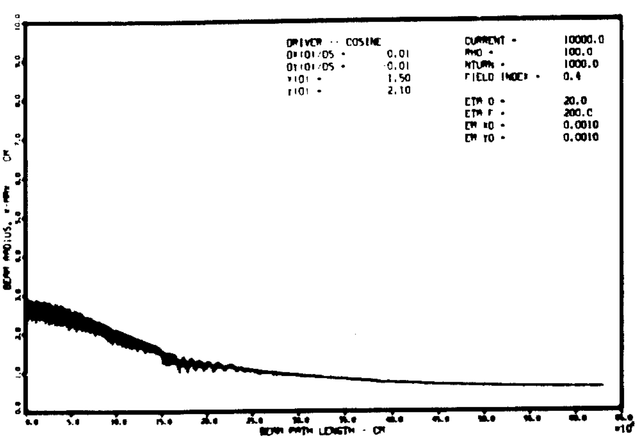

Fig. V. X-Profile for 10 kA, $X_0' = Y_0' = -10$ mrad

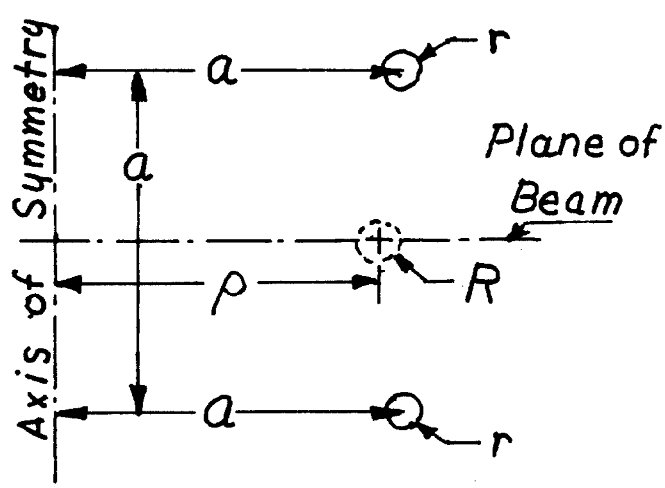

Fig. VI. Helmholtz-Coil Geometry

$$\frac{M'}{L_1'} = k_1 \leq 1 \qquad \frac{M'}{L_2'} = k_2 \leq 1 \tag{9}$$

and the inductances

$$L_1 = N_1^2 L_1' \qquad L_1 = N_2^2 L_2' \qquad M = N_1 N_2 M' \tag{10}$$

The Transformer Analogy

When it is treated as a zero-resistance system, one initially finds the following simple relations between primary voltage E_1 and current I_1, and secondary voltage E_2 and current I_2.

$$E_2 = -\frac{M}{L_1} E_1 = -k_1 \frac{N_2}{N_1} E_1 \tag{11}$$

$$I_2 = -\frac{M}{L_2} I_1 = -k_2 \frac{N_1}{N_2} I_1 \tag{12}$$

with a concomitant power-transfer efficiency of $k_1 k_2$. Equation (11) shows that the voltage gain per turn of the secondary is limited by the voltage gain per turn on the primary

$$\frac{E_2}{N_2} = -k_1 \frac{E_1}{N_1}$$

and thus in effect by the quality of the primary insulation. The importance of maximizing k_1 is obvious, because a large voltage gain per turn will minimize the number of secondary turns and thus reduce synchrotron radiation losses. When k_1 and k_2 are calculated for the conditions: $\rho = 1.0$ m, $a = 1.1$ m, $r = R = 1.0$ cm, and $f = 1$, the coupling constants are:

$k_1 = 0.507$, $k_2 = 0.152$

The k_1 and k_2 values are almost scale-independent for the same geometry, so if a maximum primary voltage per turn is assumed to be 100 kV, the maximum beam voltage gained per turn is about 50 kV in any air-core betatron.

A detailed calculation of the transformer circuit, with resistances, yields the following expression for E_2 and thus for the voltage actually applied to the beam, E_b,

$$E_b = E_2 - I_2R_2 = -k_1 \frac{N_2}{N_1}(E_1 - I_1R_1) \tag{13}$$

in which R_1, R_2 are, respectively, the resistances on the primary and secondary sides. When the total energy delivered to the beam is calculated, it is realized that the peak beam current, I_b, is related to I_2 by this expression:

$$I_2 = \frac{I_b f}{N_2} \tag{14}$$

and R_2 to R_b by the inverse ratio,

$$R_2 = \frac{N_2 R_b}{f} \tag{15}$$

where R_b is of the order of $10^{-3}\Omega$.[3] When an estimate of the primary current I_1 is made, using Eqs. (12) and (14), it is found that it has the magnitude:

$$I_1 \simeq -\frac{I_b}{k_2 N_1} \tag{16}$$

So I_1 must be of the order of ($3.3\ I_b$) for a two-turn primary, divided equally between the turns. Substantial voltage loss can occur in the primary unless the primary resistance together with its power supply have very low impedance.

Synchrotron Radiation and Maximum Energy

This has not been included in the previous sections and is in general difficult to include in simple form, because the equation for the relativistic energy gain per turn:

$$\frac{d\gamma}{dN_2} = \frac{eE_b(N_2)}{m_e c^2} - \frac{4\pi}{3}\frac{r_e}{\rho}\gamma 4 \tag{17}$$

is in general resistant to analytic solution. In general, depending on the time profile of E_1, E_b varies with the turn number, N_2. The question of a maximum γ can be simply resolved, however. If it occurs the first derivative of γ with respect to N_2 must be zero, and its second derivative less than zero. But imposition of both conditions requires:

$$\frac{d^2\gamma}{dN_2^{\ 2}} = \frac{e}{m_e c^2}\frac{dE_b}{dN_2} < 0$$

and this is a situation which can be avoided if E_b continues to increase with N_2 (time) during the entire accelerating cycle. There is thus in principle no theoretical upper limit to the output energy of a betatron. But practical limitations exist, one of which has been noted. If only 50 kV/turn can be obtained, Eq. (17) suggests a maximum energy of about 866 MeV for a ρ of 1.0 m, and this limit increases only as $(\rho)^{1/4}$. This would moreover be an extremely inefficient accelerator, with perhaps half of the delivered power going into sychrotron radiation. (The exact fraction would depend on the time profile of E_b.) An estimated 7×10^4 turns would be required, so that about 1.5 ms would be needed to deliver the power. Thus, for $I_b = 1$ kA, the average power required during the acceleration time would be about 23 TW, using the cited power factor of k_1k_2. The amount of stored energy must be of the order of 35 GJ.

Comment

It does not appear that a true air core or "ironless" betatron is a practical accelerator. However, for high energy, the inclusion of some iron to guide magnetic fields could also act to increase the coupling constants k_1 and k_2. Additionally, the efficiency should be examined as a function of the time profile of E_b. There is probably some "most efficient" form of this profile for each desired final energy and beam radius, because of losses to synchrotron radiation.

1. Kapchinskii, I.M., and Vladimirskii, V.V., Proc. Int. Conf. on High Energy Accelerators, CERN, Geneva, p. 274.
2. Courant, E.D. and Snyder, H.S., Ann. Phys 3, 1 (1958).
3. Lawson, J.D., "The Physics of Charged-Particle Beams," Clarendon (1977) Eq. 6.83, p. 334.