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NORMAL OSCILLATING MODES OF A ONE-DIMENSIONAL MAXWELLIAN BEAM IN THE LINEAR SPACE CHARGE APPROXIMATION

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Abstract

Following a method used in plasma physics [1], a Fourier-Hermite transform is attempted to solve the linearized Vlasov-Poisson equation for small fluctuations of a stationary beam. This expansion method is well fitted to the case of a maxwellian beam confined by linear external forces. As a first approximation, if one retains only the linear part of the space charge forces, the eigenmodes for the small beam oscillations are found in a straightforward way. The associated eigenfrequencies are the roots of a 3-diagonal determinant. The spectrum involves only real and discrete eigenfrequencies, and is similar to the spectrum of a beam with uniform density in real space [2] or in phase space [3].

We consider a one dimensional proton beam with uniform longitudinal velocity and one degree of freedom in the transverse direction. The particles are focused by a linear external force acting against the electric self-force and obey single particle equations [4] such as

$$\frac{dx}{dt} = u \qquad \frac{du}{dt} = \omega_p^2 \varepsilon - \nu_o^2 x \quad . \tag{1}$$

Then the Vlasov-Poisson equation for the beam density f(x,u,t) in phase space has the following form

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + (\omega_p^2\varepsilon - \nu_o^2 x)\frac{\partial f}{\partial u} = 0 \qquad \frac{\partial \varepsilon}{\partial x} = \int_{-\infty}^{+\infty} f(x,u,t)\,du \quad . \tag{2}$$

Here $\omega_p^2 = \frac{e^2}{\varepsilon_o m}$ is the central plasma frequency (x=0) provided that the central density in real space equals unity. The stationary solutions of Eqs (2) are arbitrary functions F of the energy

$$W = \frac{1}{2}(\nu_o^2 x^2 + 2\omega_p^2\phi + u^2) \qquad \varepsilon = -\frac{d\phi}{dx} \quad . \tag{3}$$

The stationary maxwellian distribution of interest is therefore

$$F(W) = \frac{1}{\pi^{1/2}} \exp(-2W) \quad , \tag{4}$$

while the stationary density $F_o(x)$ in real space is determined by the following equations, as a result of Eqs (2,3,4) :

$$\frac{d^2}{dx^2}(\text{Log } F_o) + 2(\nu_o^2 - \omega_p^2 F_o) = 0 \qquad F_o(x) = \exp -(\nu_o^2 x^2 + 2\omega_p^2\phi) \tag{5}$$

An increasing space-charge intensity shifts the real density $F_o(x)$ more and more away from a gaussian configuration. As a consequence the non-linear part of the self-force causes a spread $\nu(W)$ of the particle frequencies along the trajectories W = constant in phase space. Infinitely small amplitude oscillations have the limiting frequency

$$\nu = \nu(W=0) = (\nu_o^2 - \omega_p^2)^{1/2} \tag{6}$$

I. The linearized Vlasov-Poisson equation

Assuming now that the density f(x,u,t) allows small time dependent fluctuations g around the stationary distribution F such as

$$f(x,u,t) = F(W)\left(1 + g(x,u)\exp(j\omega t)\right), \qquad \varepsilon(x) + E(x,t) = \int_{-\infty}^{+\infty} f(x,u,t)\,du \quad , \tag{7}$$

and the linearized Vlasov-Poisson Eq. (2) for g may be written as

$$j\omega g + u\frac{\partial g}{\partial x} - \frac{\partial W}{\partial x}\frac{\partial g}{\partial u} - 2j\frac{\omega_p^2}{\omega} u \int_{-\infty}^{+\infty} Fg\,du = 0 . \tag{8}$$

In Eq. (8) the Poisson equation for the electric field perturbation E has been replaced by the Maxwell equation

$$j\omega E = -\int_{-\infty}^{+\infty} Fg\,du \quad . \tag{9}$$

The ω are eigenfrequencies to be found, and for that purpose we look for solutions of Eq. (8,9) having the form of an Hermite polynomial expansion :

$$g(x,u) = \sum_{k=0}^{\infty} g_k(x)H_k(u) \qquad g_k(x) = \frac{1}{\pi^{1/2}2^k k!}\int_{-\infty}^{+\infty} \exp(-u^2)g(x,u)H_k(u)\,du \tag{10}$$

with the orthogonality relation

$$\int_{-\infty}^{+\infty} \exp(-u^2)H_k H_\ell\,du = \delta_{k\ell}\pi^{1/2}2^k k!$$

Multiplying Eq. (8) by $\exp(-u^2)H_k(u)$, and integrating over the velocity space yields an infinite system of coupled equations for g_k :

$$\frac{1}{2}\frac{dg_{k-1}}{dx} + j\omega g_k - j\delta_{k1}\frac{\omega_p^2}{\omega}F_o g_k + (k+1)\left(\frac{dg_{k+1}}{dx} - 2\frac{\partial W}{\partial x}g_{k+1}\right) = 0 \quad , \tag{11}$$

while Eq. (9) is transformed into

$$j\omega E = -F_o g_1 \qquad (12)$$

After a substitution of g_o from Eq. for k = 0 into the Eq. for k = 1, the low order equations (11) become finally

$$k = 0$$

$$j\omega g_o + \frac{dg_1}{dx} - 2\frac{\partial W}{\partial x} g_1 = 0$$

$$k = 1$$

$$\frac{d^2 g_1}{dx^2} - 2\frac{\partial W}{\partial x}\frac{dg_1}{dx} + 2g_1(\omega^2 - \nu_o^2) - 4j\omega\left(\frac{dg_2}{dx} - 2\frac{\partial W}{\partial x} g_2\right) = 0 \qquad (13)$$

$$k \geq 2$$

$$\frac{1}{2}\frac{dg_{k-1}}{dx} + j\omega g_k + (k+1)\left(\frac{dg_{k+1}}{dx} - 2\frac{\partial W}{\partial x} g_{k+1}\right) = 0.$$

These are the exact linearized equations for g where all space charge effects are now included in the unique force-term $\frac{\partial W}{\partial x}$. Unfortunately, elementary solutions put severe constraints on the forces.

II. Linear space charge contribution to the modes

Keeping only the linear part of the self-forces, one has :

$$\frac{\partial W}{\partial x} = (\nu_o^2 - \omega_p^2)x = \nu^2 x \qquad . \qquad (14)$$

In that case a cut-off of the infinite system of Eq. (13) above the order k = m ≥ 1 occurs as a consequence of the existence of polynomial solutions such as

$$g_k(\xi) = h_k(\omega,\nu) H_{m-k}(\xi) \quad ; \quad k = 0,1,\ldots m \qquad (15)$$

with

$$\xi = \nu x \qquad .$$

The differential system (13) becomes algebraic and takes the form

$$[M_m]\,(h_k) = 0 \quad , \quad (h_k) = (h_o, h_1, \ldots h_m)$$

where $[M_m]$ is the three-diagonal matrix

$$[M_m] = \begin{bmatrix} j\frac{\omega}{\nu} & -1 & 0 & & \\ m & j\frac{\omega}{\nu}\epsilon_p & -2 & & \\ 0 & m-1 & j\frac{\omega}{\nu} & -3 & \\ & & m-k+1 & j\frac{\omega}{\nu} & -(k+1) \\ & & & \ddots & \ddots \\ & & & 0 & 1 \quad j\frac{\omega}{\nu} \end{bmatrix} \qquad (16)$$

with $\epsilon_p = 1 - \omega_p^2/\nu^2$

The eigenfrequencies up to order m are therefore the roots of

$$|M_m| = \mathrm{DET}\,[M_m] = 0 \qquad , \qquad (17)$$

and looking at the space charge contribution to the modes, Eq. (17) separates according to

$$|M_m|_{\omega_p^2=0} + \frac{\omega_p^2}{\nu^2}\,|M'_m| = 0 \qquad , \qquad (18)$$

where $|M'_m|$ is the determinant of the matrix between dashed brackets in (16) for m ≥ 2, and $|M'_1| = 1$. Finaly the eigenfrequencies ω are defined as the roots of

$$K_m(\omega) = \frac{\omega_p^2}{\nu^2}\,\frac{|M'_m|}{|M_m|_{\omega_p^2=0}} = -1 \qquad . \qquad (19)$$

The first K_m's are

$$K_1 = \frac{\omega_p^2}{\nu^2}\,\frac{1}{(1 - \omega^2/\nu^2)} \quad ; \quad K_2 = \frac{\omega_p^2}{\nu^2}\,\frac{1}{(2^2 - \omega^2/\nu^2)},$$

with the recursion relation

$$K_m = \frac{\omega_p^2/\nu^2 + (m-1)(m-2)K_{m-2}}{(m^2 - \omega^2/\nu^2)} \qquad (20)$$

Eq. (19) gives one eigenfrequency with m = 1,2, two with m = 3,4, three with m = 5,6, etc, and they are usually labelled ω_{mn}, n = m,m-2,m-4,.... (n is the azimuthal harmonic number and m the radial node number of the mode in phase space). The first ω_{mn}'s are shown in Figure 1 in terms of ω_p^2/ν_o^2. The associated eigenfunctions for small perturbations up to order m are

$$F(w) g_{mn} = \exp -(\xi^2 + u^2)\exp(j\omega_{mn} t) \times \sum_{k=0}^{m} h_k(\nu) H_{m-k}(\xi) H_k(u) \qquad . \qquad (21)$$

III. Low beam intensities

In this case the eigenfrequencies ω_{mn} are only slightly shifted up from the harmonic frequencies nν and have approximately the form

$$\omega_{mn} = n\nu + \frac{\lambda_{mn}}{n}\,\Delta\nu \qquad (22)$$

$$\Delta\nu = \nu_o - \nu = \omega_p^2/2\nu_o$$

(Δν is the maximum frequency spread between central and external trajectories in phase plane). The solution

of Eq. (19) to first order with respect to $\Delta\nu/\nu$ gives

$$\lambda_{mn} = \frac{n^2(m-1)!}{2^{m-1}\left(\frac{m-n}{2}!\right)\left(\frac{m+n}{2}!\right)} \quad (23)$$

We notice that all eigenfrequencies with the same harmonic number n are clustered between $n\nu$ and $n\nu + \frac{\Delta\nu}{n}$ (Figure 2). The first order frequency shift (22,23) agrees with the perturbation theory, which is connected with the orthogonality of the modes, and the associated eigenfunctions (21) become Laguerre polynomials as a function of the radius in phase-space [5].

Conclusion

For practical low beam intensities ($\omega_p^2 << \nu_o^2$) the eigenfrequency shifts from harmonics $n\nu$, due to linear space charge effects, are of some importance for the lower modes only. One has for instance a rigid dipole mode with the unperturbed frequency $\omega_{11} = \nu_o$ independent of intensity, probably also unaffected by the non-linear space charge and undamped according to the first-moment equations. But more generally, it is thought that the non-linear space charge forces interact with the above modes and determine a more realistic behaviour of oscillating beams. Standard methods [6] are attempted at present, as well as a numerical simulation of the oscillating modes for comparison, so as to estimate the stability of the lower modes.

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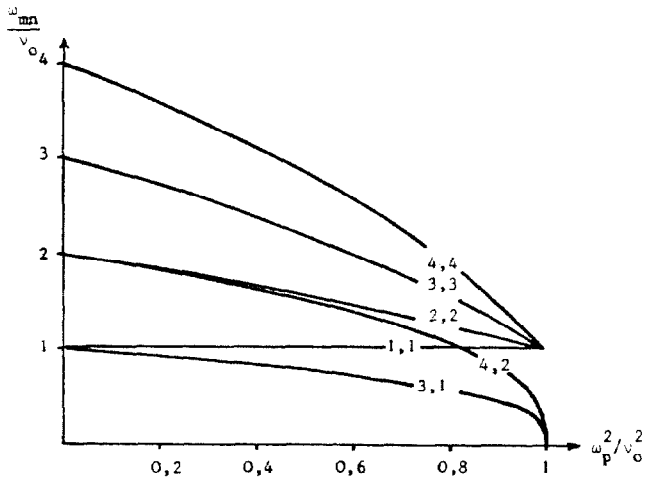


Figure 1 : Linear space charge contribution to the low-order eigenfrequencies $m \leq 4$

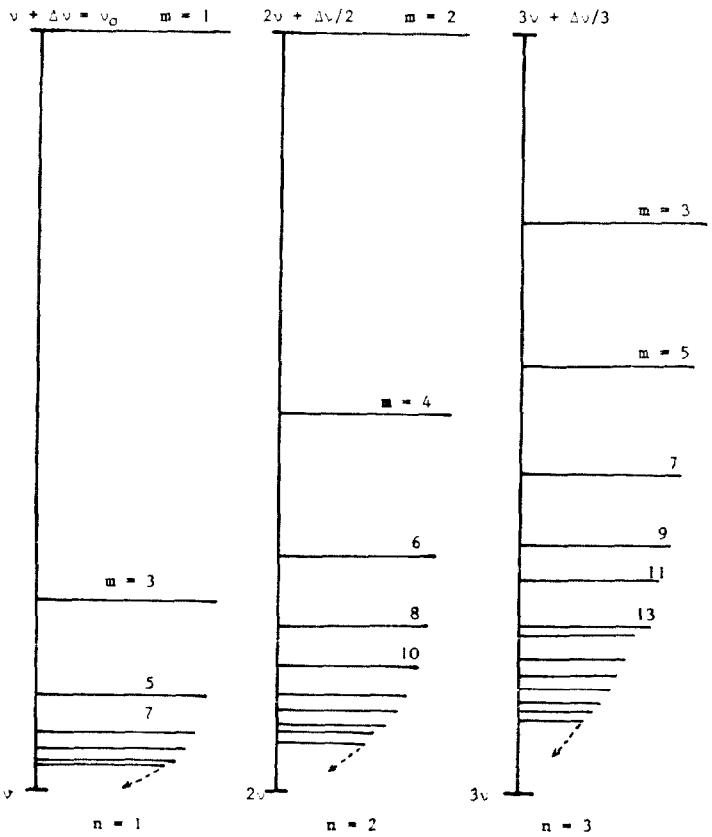


Figure 2 : Clustered eigenfrequencies with harmonic number n = 1,2,3 for low beam intensities.