COHERENT SYNCHROTRON RADIATION AND WAKE FIELDS WITH DISCONTINUOUS GALERKIN TIME DOMAIN METHODS

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Outline of Talk

- Motivation for the Study of Coherent Synchrotron Radiation (CSR)
- Maxwell's Equations + Transformations
- Brief Summary of Discontinuous Galerkin (DG)
- Simulations of Wake Fields and CSR
  - Test Case: tapered rectangular chamber
  - Bunch Compressor Case: CSR in model of DESY BC0
- Summary and Future Outlook

Motivation

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- Study the generation and propagation of CSR
- Some approximations:
  - Ultra-relativistic electron bunch ($\beta = 1$) along a curved planar orbit (2D orbit)
  - Consider rectangular cross-section vacuum chambers such as in a bunch compressor (2D domain)
  - Consider all boundaries as PEC (open ends possible)
  - Ignore collective effects for this work (known source terms)
- Goals:
  - Compute electromagnetic fields in the domain
  - Compute longitudinal wake potential

Maxwell's Equations and Coordinates

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- Maxwell's Equations
  - Starting with Cartesian coordinates: $\mathbf{R} = (Z, X, Y), \quad \tau = ct$

$$\nabla \times \mathbf{E} = -Z_0 \frac{\partial \mathbf{H}}{\partial \tau}, \qquad \nabla \times \mathbf{H} = \frac{1}{Z_0} \frac{\partial \mathbf{E}}{\partial \tau} + \mathbf{j}$$

  - Next consider a planar reference orbit (along $Y = 0$):

$\mathbf{R}_r(s) = (Z_r(s), X_r(s), 0)$ parameterized by arc length s

  - Define curvilinear coordinate transformation by:

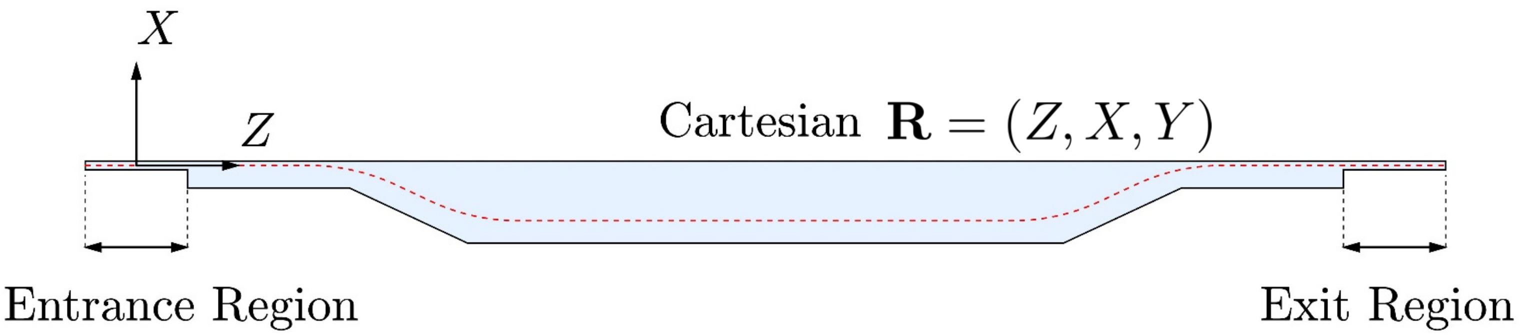
$$\mathbf{e}_s = (Z_r'(s), X_r'(s), 0), \quad \mathbf{e}_x = (-X_r'(s), Z_r'(s), 0), \quad \mathbf{e}_y = (0, 0, 1)$$

  - Also, define signed curvature κ and scale factor η by:

$$\kappa(s) = Z_r''(s) X_r'(s) - Z_r'(s) X_r''(s), \qquad \eta(s, x) = 1 + \kappa(s) x$$

Example of Geometry and Coordinates

- Example of mapping to curvilinear coordinates:



- **Advantage**: source orbit is straight: simple modeling of source with small transverse size with DG
- **Disadvantage**: only works if $\eta > 0$, problems with large κ

Source Term Definitions

- Charge and current model for ultra-relativistic bunch
  - In (s, x, y) coordinates: $\rho(s, x, y, \tau) = q\lambda(s - \tau)\delta(x)G(y)$

    $\mathbf{j}(s, x, y, \tau) = qc\lambda(s - \tau)\delta(x)G(y)\mathbf{e}_s$

    with Gaussian distributions: $\lambda(s),\ G(y)$

    and Dirac distribution: $\delta(x)$
  - Note: $\sigma_s,\ \sigma_y$ for $\lambda(s),\ G(y)$ chosen such that source terms are supported only in the entrance region at $\tau = 0$
  - Other distributions can be used – can be coupled to a particle tracking code in the future

Fourier Series Decomposition

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- Domain with parallel planar walls: $y = \pm h/2$
  - Assuming PEC boundaries: use the Fourier series

$$f(s,x,y,\tau) = \sum_{p=1}^{\infty} f_p(s,x,\tau)\phi\left(\alpha_p(y+h/2)\right),$$

$$f_p(s,x,\tau) = \frac{2}{h}\int_{-h/2}^{h/2} f(s,x,y,\tau)\phi\left(\alpha_p(y+h/2)\right)dy,$$

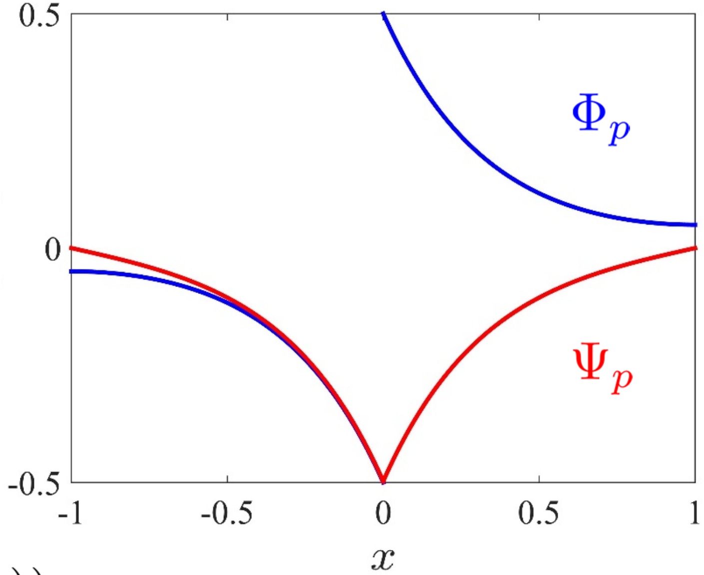
$$\alpha_p = \pi p/h, \qquad \phi(\cdot) = \sin(\cdot) \text{ or } \cos(\cdot)$$

  - E_s, E_x, H_y, j_s, j_x use sine series and E_y, H_s, H_x, j_y use cosine
  - If source is symmetric about $y=0$ then even modes vanish
  - If $\sigma_y \ll h$, more Fourier series terms required

Initial Conditions

- With PEC boundary conditions for $a \leq x \leq b$

$$E_{sp}(s,x,0) = 0$$
$$E_{xp}(s,x,0) = -qZ_0 c G_p \lambda(s) \Phi_p(x)$$
$$E_{yp}(s,x,0) = -qZ_0 c G_p \lambda(s) \Psi_p(x)$$
$$H_{sp}(s,x,0) = 0$$
$$H_{xp}(s,x,0) = qcG_p \lambda(s) \Psi_p(x)$$
$$H_{yp}(s,x,0) = -qcG_p \lambda(s) \Phi_p(x)$$



$$\Phi_p(x) = \sinh(\alpha_p b)\frac{\cosh(\alpha_p(x-a))}{\sinh(\alpha_p(b-a))} - \cosh(\alpha_p x)\Theta(x)$$

$$\Psi_p(x) = \sinh(\alpha_p b)\frac{\sinh(\alpha_p(x-a))}{\sinh(\alpha_p(b-a))} - \sinh(\alpha_p x)\Theta(x)$$

Combining All Transformations

- Issue: how to evaluate $\delta(x)$ in $\partial E_{sp}/\partial\tau$ equation?
- Fix: replace H_{yp} by $\tilde{H}_{yp} = H_{yp} - qcG_p\lambda(s-\tau)\Theta(x)$
- Result:
  - ✓ Maxwell's Eqs.
  - ✓ C. Transform
  - ✓ Source Def.
  - ✓ F. Decomp.
  - ✓ Smoother Src.
  - ✓ Initial Conds.

$$\frac{1}{Z_0}\frac{\partial E_{sp}}{\partial\tau} = \frac{\partial\widetilde{H}_{yp}}{\partial x} + \alpha_p H_{xp}$$

$$\frac{1}{Z_0}\frac{\partial E_{xp}}{\partial\tau} = -\alpha_p H_{sp} - \frac{1}{\eta}\frac{\partial\widetilde{H}_{yp}}{\partial s} - \frac{1}{\eta}qcG_p\lambda'(s-\tau)\Theta(x)$$

$$\frac{1}{Z_0}\frac{\partial E_{yp}}{\partial\tau} = \frac{1}{\eta}\frac{\partial H_{xp}}{\partial s} - \frac{\partial H_{sp}}{\partial x} - \frac{\kappa}{\eta}H_{sp}$$

$$Z_0\frac{\partial H_{sp}}{\partial\tau} = \alpha_p E_{xp} - \frac{\partial E_{yp}}{\partial x}$$

$$Z_0\frac{\partial H_{xp}}{\partial\tau} = \frac{1}{\eta}\frac{\partial E_{yp}}{\partial s} - \alpha_p E_{sp}$$

$$Z_0\frac{\partial\widetilde{H}_{yp}}{\partial\tau} = \frac{\partial E_{sp}}{\partial x} + \frac{\kappa}{\eta}E_{sp} - \frac{1}{\eta}\frac{\partial E_{xp}}{\partial s} + qZ_0cG_p\lambda'(s-\tau)\Theta(x)$$

Brief Summary of DG

- Discontinuous Galerkin Finite Element Method
  - General idea: approximate each field by an $N\mathrm{th}$ order polynomial in (s, x) on a triangular element
  - Fields are not imposed to be continuous across edges
  - Derivatives are computed for each element independently
  - Elements are coupled with a flux function
  - Naturally handles discontinuities if they exist along element edges such as $\Theta(x)$ along $x = 0$
  - Great for hyperbolic problems and easily parallelized
  - Well-suited for GPU computing since operators are dense

Final Steps for the Numerical Method

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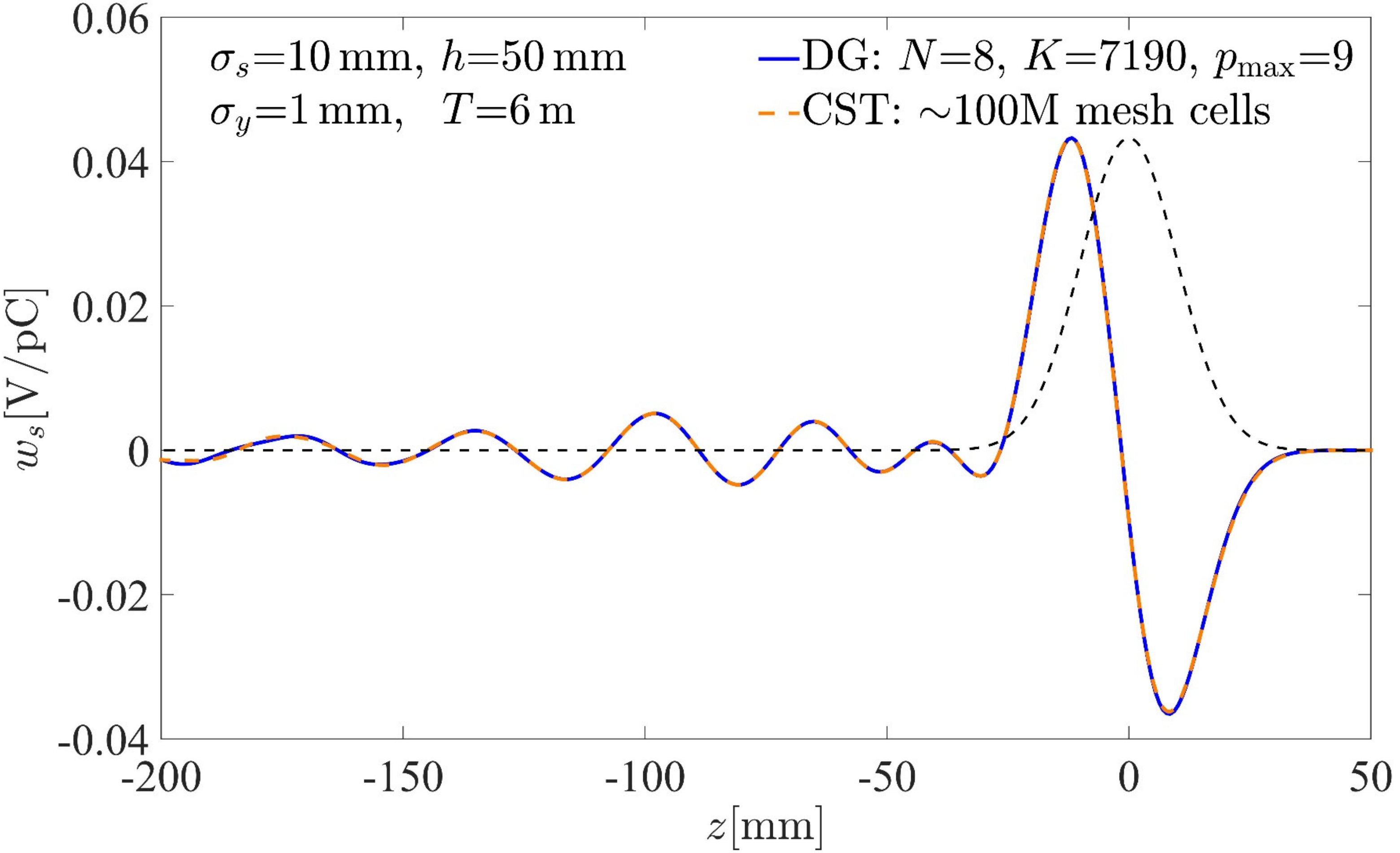
- Additional Notes:
  - Evolve fields with 4th order low-storage RK
  - Important: align elements along $x = 0$ and where κ is discontinuous (i.e. when using piecewise-defined orbits)
  - Sum over p modes for full 3D solution
  - Solution can easily be remapped to Cartesian coordinates
  - Define longitudinal wake function by:

$$w_s(z) = \frac{-1}{q} \int_0^T E_s(\tau - z, 0, 0, \tau) d\tau$$
$$= \frac{-1}{q} \sum_{p=1}^{p_{\max}} \sin\left(\frac{\pi p}{2}\right) \int_0^T E_{sp}(\tau - z, 0, \tau) d\tau$$

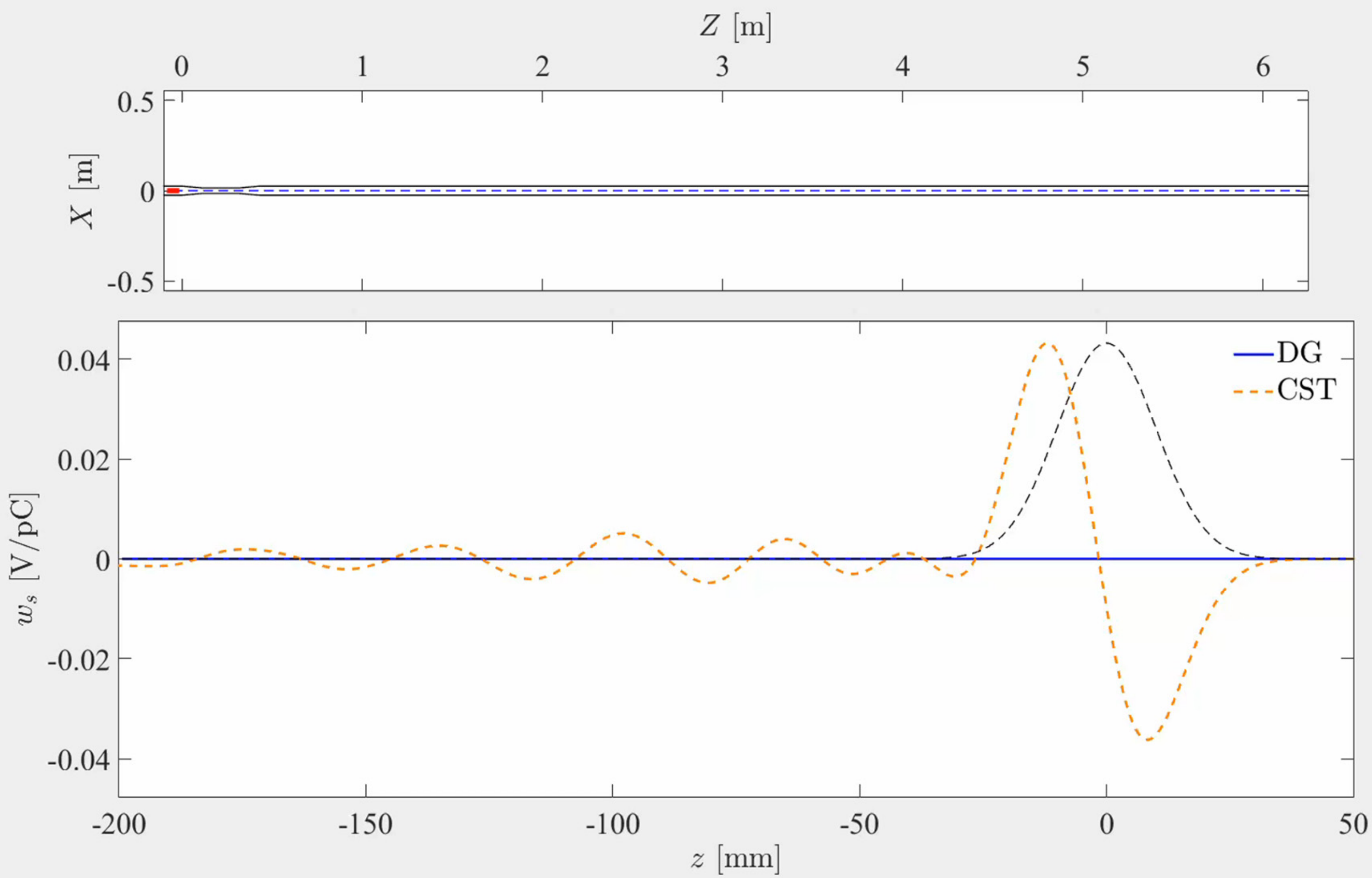
Test Case Simulation

- Test Case
  - Straight wave-guide with a taper
  - Geometry generates wake
  - E_{sp} sampled along $x = 0$ near source, sum over $p = 1, ..., 9$
  - Comparison to solution from CST Particle Studio™

Test Case Wake Function

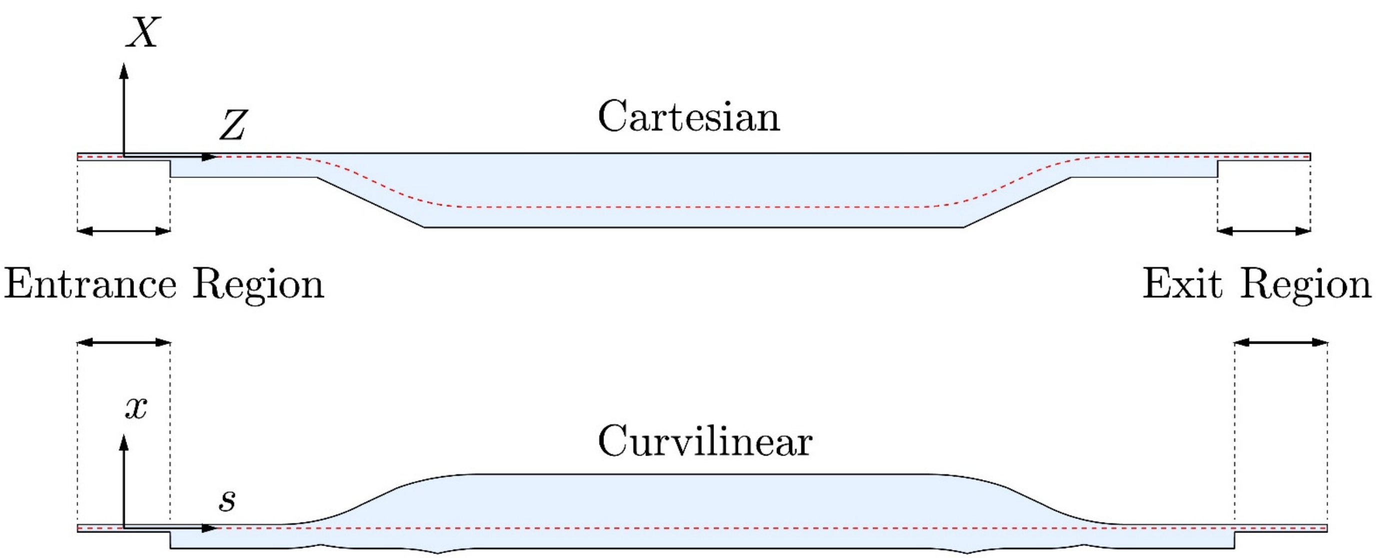


Test Case Wake Simulation

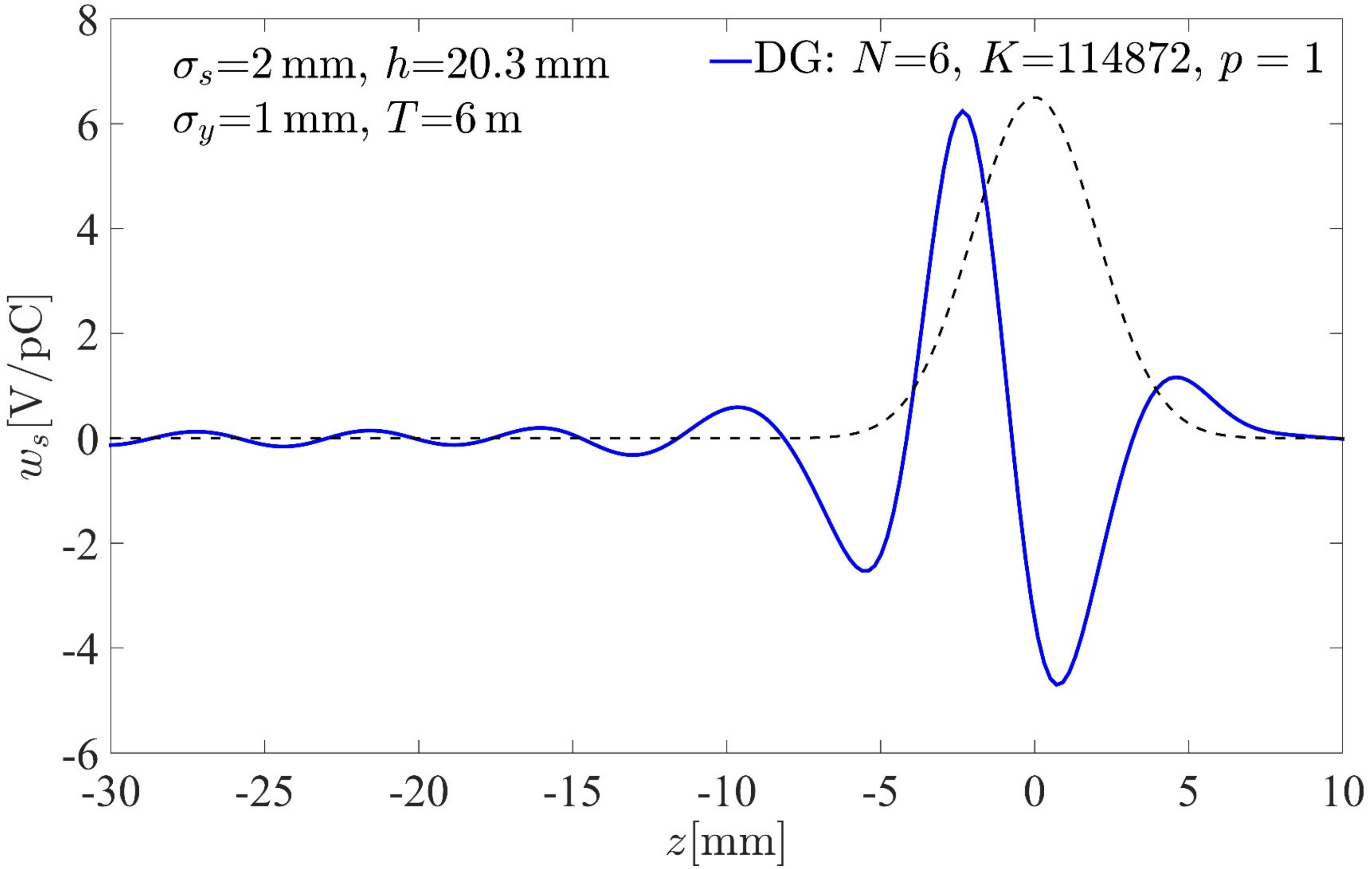


Bunch Compressor Simulation

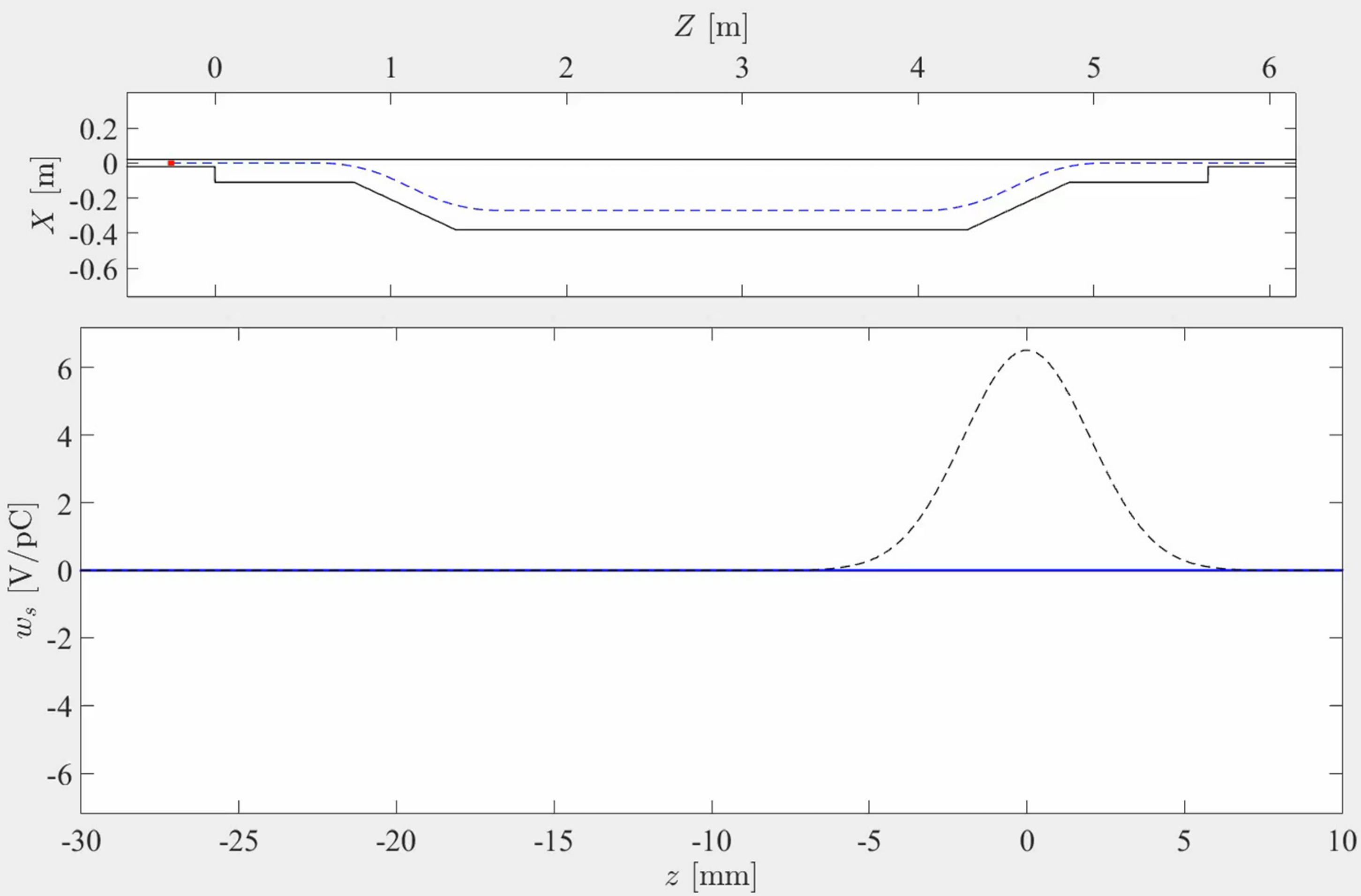
- Bunch Compressor Case
  - DESY BC0 – assume piecewise constant curvature
  - CSR and geometry generates wake
  - E_{sp} sampled along $x = 0$ near source



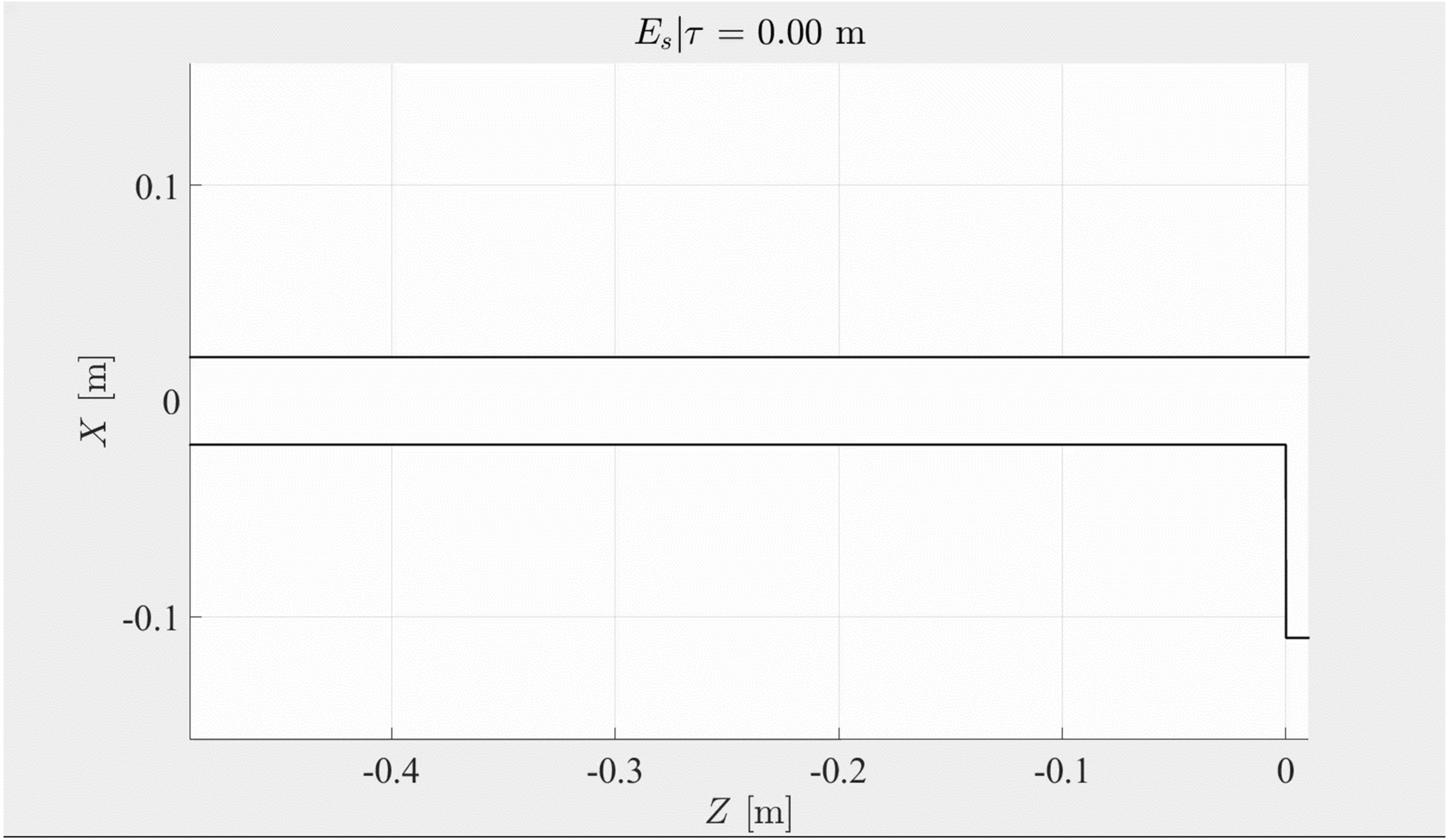
Bunch Compressor Wake Function



Bunch Compressor Wake Simulation



Bunch Compressor Field Simulation



Summary and Future Outlook

- Developed 2D time-domain field solver for ultra-relativistic bunches with a curvilinear transformation and Fourier decomposition using DG
- GPU-enabled MATLAB code built on DG Methods from "Nodal Discontinuous Galerkin Methods" by *J. Hesthaven, T. Warburton*
- Future Outlook:
  - Couple computed wake fields with particle tracking
  - Estimate wall losses with Poynting flux on walls
  - Additional validation with other EM field or CSR codes

Thank you for your attention!

Work suppported by:

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