MULTIPOLE FIELD EFFECTS IN A TRANSVERSE GRADIENT UNDULATOR

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Abstract

Using a transverse gradient undulator (TGU) is one of the methods proposed in order to enable the utilization of electron beams with large energy spread (such as those from plasma-based accelerators) in a free-electron laser (FEL). Most of the analytical treatments of this scheme assume a linear variation of the undulator field with one of the transverse coordinates. While this assumption leads to a simplified and more tractable model, including higher-order multipoles allows us to offer a more complete and rigorous description of the system. In this paper, we investigate the magnetic field components of a TGU using both theory and simulation and explore the impact of higher-order multipoles on the FEL performance.

INTRODUCTION

Due to their high brightness (a combination of high peak current and low transverse emittance), the electron beams from modern plasma-based accelerators are considered attractive options for driving compact X-ray free-electron lasers. However, one of the main inhibiting factors is the relatively large energy spread that typically characterizes such beams. One possible solution involves the use of a transverse gradient undulator (TGU), in which the undulator poles are canted in such a way as to induce a dependence of the vertical field amplitude with the horizontal coordinate [1]. Since this dependence is assumed to be predominantly linear, the resulting transverse field gradient can be used to mitigate the effect of the energy spread on the lasing, provided a suitable horizontal dispersion is added to the beam. A number of recent studies have explored the potential of such a high-gain, TGU-based FEL using both self-consistent theory and simulation [2, 3]. However, some authors [4] have expressed scepticism about some of the assumptions used in these studies, in particular those having to do with the linear character of the undulator field. Motivated by these concerns, we seek to extend the standard treatment of the subject by explicitly considering the impact of nonlinear field components on the operation of a TGU-based FEL.

FIELD ANALYSIS

The main objective of this section is to present a multipole analysis of the magnetic field of a transverse gradient undulator (TGU) from an analytical point of view. The magnetic field $\mathbf{B}_u$ satisfies the vacuum Maxwell equations $\nabla\times\mathbf{B}_u = 0$ and $\nabla\cdot\mathbf{B}_u = 0$. Thus, it can be derived from a scalar magnetic potential Φ_m via $\mathbf{B}_u = \nabla\Phi_m$, where the potential itself satisfies the Laplace equation ($\nabla^2\Phi_m = 0$). Assuming a simple harmonic z-dependence for the scalar potential (i.e. $\Phi_m = \Phi(x, y)\sin(k_u z)$, where $k_u = 2\pi/\lambda_u$ and λ_u is the undulator period), one easily finds that the potential amplitude Φ satisfies

$$\nabla_\perp^2\Phi - k_u^2\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi - k_u^2\Phi = 0\,. \qquad (1)$$

The field components are given by

$$\begin{aligned} B_{ux} &= \frac{\partial\Phi_m}{\partial x} = B_x(x, y)\sin(k_u z)\,,\\ B_{uy} &= \frac{\partial\Phi_m}{\partial y} = B_y(x, y)\sin(k_u z)\,,\\ B_{uz} &= \frac{\partial\Phi_m}{\partial z} = B_z(x, y)\cos(k_u z)\,, \end{aligned} \qquad (2)$$

where $B_x = \partial\Phi/\partial x$, $B_y = \partial\Phi/\partial y$ and $B_z = k_u\Phi$. Note that the three amplitudes B_i (where $i = x, y, z$) also satisfy the same equation as the potential, i.e. $\nabla_\perp^2 B_i - k_u^2 B_i = 0$. Our goal is to determine the field's full transverse dependence given the vertical field B_y at the median plane $y = 0$ (that is, $B_y(x, 0)$). Thus, we assume that the latter is known and given in a Taylor series form by

$$B_y(x, 0) = \sum_{i=0}^{\infty} F_i x^i = F_0 + F_1 x + F_2 x^2 + \ldots\,, \qquad (3)$$

where the F_i coefficients will be determined in due course. Moreover, we seek a separable solution of the form $\Phi(x, y) = X(x)Y(y)$. This leads to the relation $X''/X + Y''/Y = k_u^2$ (here, the double prime denotes the second derivatives d^2/dx^2 and d^2/dy^2). This, in turn, implies that $X'' = k_x^2 X$ and $Y'' = k_y^2 Y$, where k_x and k_y are constants that satisfy $k_x^2 + k_y^2 = k_u^2$. The resulting solution is

$$\begin{aligned} \Phi(x, y) = \Phi_0&[\cosh(k_x x) + c_x\sinh(k_x x)]\\ &\times[c_y\cosh(k_y y) + \sinh(k_y y)]\,, \end{aligned} \qquad (4)$$

where $c_{x,y}$ and Φ_0 are constants. The corresponding vertical field is

$$\begin{aligned} B_y(x, y) = \frac{\partial\Phi}{\partial y} &= B_0[\cosh(k_x x) + \alpha_x\frac{\sinh(k_x x)}{k_x}]\\ &\times[\cosh(k_y y) + \alpha_y\frac{\sinh(k_y y)}{k_y}]\,, \end{aligned} \qquad (5)$$

where $B_0 = k_y\Phi_0$ and $\alpha_i = c_i k_i$ ($i = x, y$). This particular solution is defined in terms of two scaled gradients ($\alpha_{x,y}$) and two scaled "curvature" quantities ($k_{x,y}$). At this point, we make use of the fact that $B_y(x, -y) = B_y(x, y)$, which yields $\alpha_y = 0$. The field then reduces to

$$B_y(x, y) = B_0[\cosh(k_x x) + \alpha_x\frac{\sinh(k_x x)}{k_x}]\cosh(k_y y)\,. \qquad (6)$$

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The form previously used in [3] corresponds to setting $k_x \to 0$. The next step is to compare the median-plane field

$$B_y(x,0) = B_0[1 + \alpha_x x + \frac{(k_x x)^2}{2} + ...] \tag{7}$$

to the original expansion, i.e. $B_y(x,0) = F_0 + F_1 x + F_2 x^2 + ...$, which yields $B_0 = F_0$, $\alpha_x = F_1/B_0 = F_1/F_0$, $k_x^2 = 2F_2/B_0 = 2F_2/F_0$ and $k_y^2 = k_u^2 - k_x^2 = k_u^2 - 2F_2/F_0$. To complete the solution, we outline a procedure for calculating the original expansion coefficients F_i. When the undulator pole faces are parallel, the field has no x-dependence and the maximum, on-axis value of the vertical field (B_0) can be related to the full gap g_0 through some formula of the form $B_0 = H(g_0) = c_0 \exp(P(g_0))$, where c_0 is a constant and

$$P(g_0) = -a_1 \frac{g_0}{\lambda_u} + a_2 \left(\frac{g_0}{\lambda_u}\right)^2, \tag{8}$$

where a_1, a_2 are positive constants. For hybrid undulators, the above relation (known as Halbach's formula) is an empirical result which is valid for $g_0 < \lambda_u$ ($c_0 = 3.33$ T, $a_1 = 5.47$ and $a_2 = 1.80$ if the material used is samarium-cobalt - SmCo). For pure permanent magnet devices, a variation of this general formula (with $a_1 = \pi$ and $a_2 = 0$) can be derived analytically [5, 6]. Generalizing a previous calculation for the TGU gradient, we propose to calculate the median-plane field through the relation $B_y(x,0) \approx H(g_0 + 2\varphi x)$, where φ is the (small) TGU angle. Comparing this approximation with Eq. (3), we obtain the relation

$$F_n = \frac{H^{(n)}(g_0)}{n!}(2\varphi)^n, \tag{9}$$

where $H^{(n)}(g_0) = d^n H(g_0)/dg_0^n$. After some calculation of the derivatives involved, we find the first few coefficients (note that the double prime now denotes differentiation with respect to g_0):

$$\begin{aligned} F_0 &= B_0 = H(g_0), \\ F_1 &= 2\varphi H_0' = 2\varphi B_0 P_0', \\ F_2 &= \frac{1}{2} H_0''(2\varphi)^2 = \frac{1}{2F_0}[1 + \frac{P_0''}{P_0'^2}]F_1^2 . \end{aligned} \tag{10}$$

In view of Eq. (8), we have

$$\zeta = \frac{P_0''}{P_0'^2} = \frac{2a_2}{(a_1 - 2a_2(g_0/\lambda_u))^2} . \tag{11}$$

For $g_0 = \lambda_u$, $\zeta = 2a_2/(a_1 - 2a_2)^2 \approx 1$ but typically g_0 is considerably smaller than λ_u and $\zeta \ll 1$ (for a pure permanent magnet device, ζ is identically equal to zero). On the other hand, we expression for F_1 yields

$$F_1 = 2\varphi F_0 P_0' \to \alpha_x = \frac{F_1}{F_0} = -\frac{2\varphi}{\lambda_u}(a_1 - 2a_2 \frac{g_0}{\lambda_u}), \tag{12}$$

which is the expression given in [2]. In terms of the gradient, the coefficient for the quadratic term is rewritten as

$$F_2 = \frac{B_0}{2}(1+\zeta)\alpha_x^2 \approx \frac{B_0}{2}\alpha_x^2 \to k_x^2 = \frac{2F_2}{F_0} \approx \alpha_x^2 . \tag{13}$$

For $k_x \approx \alpha_x$, we have $B_y(x,y) \approx B_0 \exp(\alpha_x x)\cosh(k_y y)$, which is a rather useful analytical approximation.

FEL CONSIDERATIONS

To estimate the effect of the various field multipoles (i.e. contributions $\propto x^n$) on the operation of the FEL, we begin with the basic resonance condition $\lambda_r = \lambda_u(1+K^2/2)/(2\gamma^2)$, which relates the radiation wavelength λ_r to the fundamental e-beam and undulator properties. For a horizontally-dispersed beam, the relativistic factor is given by $\gamma = \gamma_0(1 + x/\eta)$, where γ_0 is the central value and η is the dispersion. Considering only the field at the median plane ($y = 0$), the undulator parameter is expressed by $K \to K(x) = eB_y(x,0)/(mck_u)$. After some rearrangement, we have

$$\lambda_r = \lambda_{r0} \frac{1 + (K^2(x) - K_0^2)/(2 + K_0^2)}{(1 + x/\eta)^2}, \tag{14}$$

where $\lambda_{r0} = \lambda_u(1 + K_0^2/2)/(2\gamma_0^2)$ and $K_0 = K(0) = eB_0/(mck_u)$. Thus, the deviation of the radiation wavelength from its on-axis value λ_{r0} is given by

$$\begin{aligned} \frac{\Delta\lambda_r}{\lambda_{r0}} &= \frac{\lambda_r - \lambda_{r0}}{\lambda_{r0}} \\ &\approx \frac{[K_0^2/(2+K_0^2)](K^2(x)/K_0^2 - 1) - 2(x/\eta) - (x/\eta)^2}{(1 + x/\eta)^2} . \end{aligned} \tag{15}$$

The above equation is valid for any field dependence $K(x)$. For a specific profile of the form $K = K_0 \exp(\alpha_x x)$, we have $K^2/K_0^2 - 1 = \exp(2\alpha_x x) - 1 = 2\alpha_x x + 2\alpha_x^2 x^2 + ...$ Thus, expanding the RHS up to second order in x, we have the expression

$$\begin{aligned} \frac{\Delta\lambda_r}{\lambda_{r0}} \approx & 2(\frac{K_0^2\alpha_x}{2+K_0^2} - \frac{1}{\eta})x \\ & + [\frac{2K_0^2\alpha_x^2}{2+K_0^2} + \frac{3}{\eta^2} - 4\frac{K_0^2\alpha_x}{2+K_0^2}\frac{1}{\eta}]x^2 . \end{aligned} \tag{16}$$

Assuming that the TGU resonance condition

$$\frac{\alpha_x K_0^2}{2 + K_0^2} = \frac{1}{\eta} \tag{17}$$

is valid and using the fact that $x/\eta \sim \sigma_\delta \ll 1$, (where σ_δ is the rms energy spread of the e-beam), we find that the scaled "detuning" quantity due to the leading (second order) field multipole is

$$\frac{\Delta\lambda_r}{\lambda_{r0}}/\rho \approx \frac{\sigma_\delta}{\rho}(1 + \frac{4}{K_0^2})\sigma_\delta , \tag{18}$$

where ρ is the FEL parameter. We point out that the RHS of the above result does not contain the dispersion η. For larger TGU angles, the gradient increases and so do the strengths of the multipole terms but the dispersion for resonant operation decreases and the reduced beam size balances the effect. For typical TGU parameters ($\sigma_\delta/\rho \sim 1$, $K_0 \sim 1$), we have $(\Delta\lambda_r/\lambda_r)/\rho \sim \sigma_\delta \ll 1$, so the multipoles should not affect the FEL appreciably.

RADIA SIMULATION

We have benchmarked our analytical expressions with the aid of the RADIA magnetostatic simulation package [7]. In particular, we considered a standard, pure permanent magnet undulator configuration with period λ_u = 14 mm, four blocks per period and a remanent field of about 1 T (the material being neodymium-iron-boron - NdFeB). In Fig. 1, we plot the various on-axis components of the magnetic field along the undulator (for a total length of 20 periods). On the other hand, Fig. 2 shows the x-dependence of the field components at the median plane ($y = 0$). In this case, the z-location is chosen so that the vertical field is maximum, in the middle of the undulator. A gap g_0 of 6 mm yields a field B_0 = 0.46 T, in line with analytical estimates [6]. For an angle φ of 12 degrees, the theoretical value for the gradient is $\alpha_x = -2\pi\varphi/\lambda_u = -94$ m^{-1}, in close agreement with the simulation value of -93.2 m^{-1} (the latter is derived from a polynomial fit of the data shown in Fig. 2), A similar agreement is observed for the second-order coefficients (the theoretical value being $\alpha_x^2/2 \approx 4417$ m^{-2} while the simulation value is 4051 m^{-2}. Even for this relatively strong gradient, Eq. (18) yields $(\Delta\lambda_r/\lambda_{r0})/\rho \sim 0.12$, for a (rather pessimistic) case with $\sigma_\delta/\rho = 1$ and $\sigma_\delta = 10^{-2}$. Even this value is mostly due to the low undulator parameter $K_0 = 0.61$ (the resonant dispersion is -68 mm). In general, the scaled detuning due to the nonlinear field components appears to be, at worst, an order of magnitude larger than the energy spread σ_δ, which is typically not more than a few percent. Thus, $(\Delta\lambda_r/\lambda_r)/\rho$ is usually much smaller than unity.

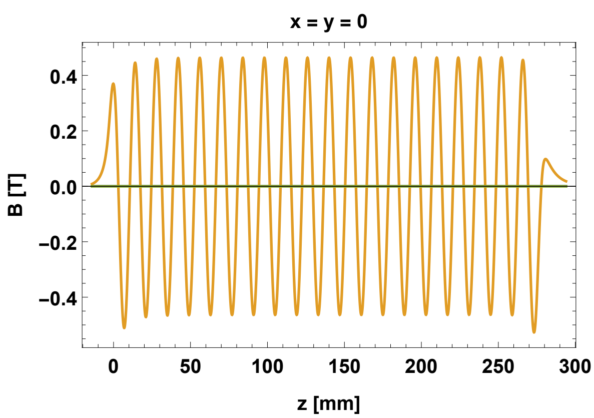

Figure 1: Longitudinal variation of the on-axis, undulator magnetic field components (RADIA simulation). The only non-zero component is the vertical field (brown line).

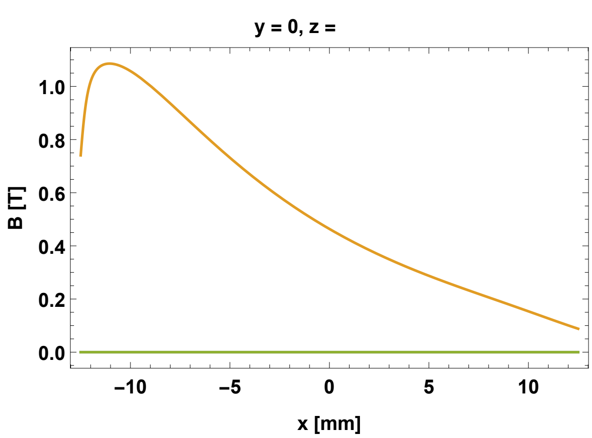

Figure 2: x-dependence of the median-plane ($y = 0$) magnetic field components (RADIA simulation). The z-location corresponds to a crest of the y-field. Once again, the brown line corresponds to the vertical field.

CONCLUSIONS

We have presented an analysis of the magnetic field components of a transverse gradient undulator (TGU), including higher-order multipoles. In particular, we have generalized previous analytical results in order to include a nonlinear dependence of the vertical field with respect to the horizontal coordinate x. Moreover, we have explored the effect of these added multipoles on the operation of a TGU-based FEL. While this investigation is by no means exhaustive, we believe that the presence of nonlinear multipoles does not introduce any insurmountable obstacles, as far as the utilization of the TGU in a compact FEL configuration is concerned.

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