SCALING LAWS FOR MAGNETIC ENERGY IN SUPERCONDUCTING QUADRUPOLES

F. Borgnolutti, E. Todesco, CERN, Geneva, Switzerland
A. Mailfert, Nancy Université, LEM, UMR 7569 CNRS-INPL, France

Abstract

The stored energy in superconducting magnets is one of the main ingredients needed for the quench calculation and for designing quench protections. Here we propose an analytical formula based on the Fourier transformation of the current density flowing within the winding to determine the magnetic energy stored in superconducting quadrupoles made of sector coils. Two corrective coefficients allowing to estimate the energy enhancement produced either by current grading or by the presence of an unsaturated iron yoke are respectively derived from a numerical and an analytical study. This approach is applied to a set of real quadrupoles to test the validity limits of the scaling law, which are shown to be of ~10%.

INTRODUCTION

Analytical computation of the magnetic stored energy in superconducting magnets is rather difficult because of the complicated geometry of the coil and it is usually done with numerical codes (see for instance [1]). However, during the conceptual design phase of a magnet it can be useful to have a simple approximated expression of the magnetic energy as a function of the main features of the magnet lay-out. Here we propose such a formula to estimate the stored energy in superconducting quadrupoles.

The study is based on the Fourier transform of the current density flowing within the coils. After deriving the stored energy formula in case of a quadrupole made of pure sector coils, we present a heuristic corrective coefficient allowing taking into account the energy enhancement due to current grading i.e coils with non-uniform current density. The impact of an unsaturated iron yoke on the stored energy is then calculated analytically. A cross-check of the formula with a numerical code shows that our approach allows to estimate the energy within 10% accuracy in all analysed cases.

PURE SECTOR COIL QUADRUPOLE

The magnetic stored energy definition used throughout this paper is

$$U_t = \frac{1}{2}\iiint_{\Omega} \boldsymbol{A}.\boldsymbol{j} d\Omega \,, \qquad (1)$$

where $\boldsymbol{A}$ is the vector potential defined by $\boldsymbol{B}=\text{curl}\boldsymbol{A}$, $\boldsymbol{j}$ is the current density flowing inside the coil, and Ω is the winding volume. The energy is in [J], the current density is in [A/m^2] and the distance in [m].

Now let us consider a quadrupole magnet made of pure sector coils (see for instance Fig. 1). Assuming that there is no non-linear magnetic material and that the coils length is infinite along the z quadrupole axis, the stored energy U per meter along z is given by:

$$U = \frac{1}{2}\iint_{S} A_z(r,\theta) j_z(r,\theta) r dr d\theta \qquad (2)$$

Where S is the cross-sectional area of the magnet.

To approach the cos2θ design, which produces a pure quadrupolar field, quadrupoles magnets are usually made of sector coils (see Fig. 1). Their geometry and positions are optimized to cancel the first allowed field harmonics.

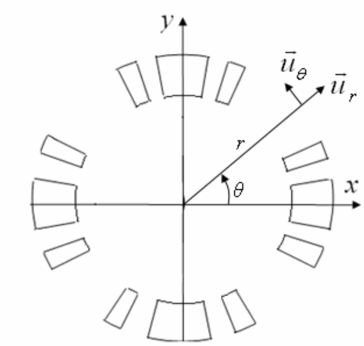

Figure 1: Cross-section of a Quadrupole magnet made of 2 pure sectors.

Lets assume a constant current density j_0 flowing inside the winding of a quadrupole made of pure sector coils. The current density can be modelled by means of the Fourier transform

$$j(\theta) = j_0 \sum_{n=1}^{\infty} a_n \cos(n\theta) \equiv \sum_{n=1}^{\infty} j_n(\theta) \,, \qquad (3)$$

where the a_n coefficients are given by:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} j(\theta)\cos(n\theta) d\theta \qquad (4)$$

For a coil made by a single sector of angular dimension θ_1 the coefficients a_n read

$$a_n = \frac{2}{n\pi}\sin(n\theta_1)\left[1+\cos(n\pi) - 2\cos\left(n\frac{\pi}{2}\right)\right] \qquad (5)$$

The vector potential produced by a pure current harmonic of rank n is given by:

$$A_{z,n}(r,\theta) = C_n(r) j_n(\theta) \qquad (6)$$

Introducing the coil inner and outer radius R_1 and R_2, the C_n coefficient is written for n=2

$$C_2(r) = \frac{\mu_0}{4}\left(\frac{1}{4r^2}\left[r^4 - R_1^{\ 4}\right] + r^2 \ln\left(\frac{R_2}{r}\right)\right), \qquad (7)$$

and for n>2,

$$C_n(r)=\frac{\mu_0}{2n}\left(\frac{1}{n+2}\left[r^2-\frac{R_1^{\,n+2}}{r^n}\right]+\frac{1}{n-2}\left[r^2-\frac{r^n}{R_2^{\,n-2}}\right]\right) \tag{8}$$

Substituting (3) and (6) in (2), and introducing the coil width $w=R_2-R_1$, one get an explicit expression for the magnetic energy coefficient

$$U_n=\frac{\pi\mu_0 j_0^{\,2}a_n^{\,2}}{4n}R_1^{\,4}f_n\left(\frac{w}{R_1}\right) \tag{9}$$

For a given cross-section layout, the ratio between the main harmonic of energy U_2 and the total energy U given by the sum of all harmonics depends only on w/R_1. The study of this ratio for several coils layouts shows that the stored energy is mainly carried by U_2 (see Fig. 2). Therefore, a good approximation for the energy stored in quadrupoles made of sector coils is:

$$U\sim U_2=\frac{\pi\mu_0 j_0^{\,2}a_2^{\,2}}{8}R_1^{\,4}f_2\left(\frac{w}{R_1}\right) \tag{10}$$

Introducing $t=w/R_1$, the function f_2 is

$$f_2(t)=\frac{1}{8}\left[(1+t)^4-1\right]-\frac{1}{2}\ln(1+t) \tag{11}$$

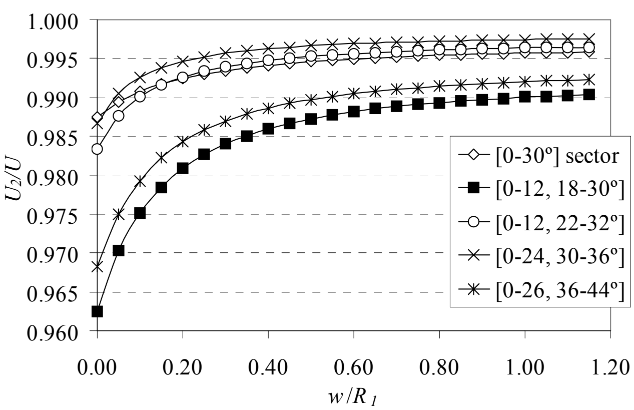

Figure 2: Ratio between the magnetic energy produced by the main harmonic (n=2) and the total energy versus w/R_1 for five different sector coils.

REAL QUADRUPOLE CROSS-SECTIONS

No Grading, No Iron Yoke

Now we aim at comparing the numerical computation of the stored energy using the numerical code [1] to the analytical estimate (10) for a set of realistic coil-layouts based on the cos2θ design. To do that, we introduce the coil width equivalent w_{eq} corresponding to the width of a pure sector coil having the same cross-sectional area S and inner radius R_1 than the real one:

$$w_{eq}=\left(\sqrt{1+\frac{3S}{2\pi R_1^{\,2}}}-1\right)R_1 \tag{12}$$

Some of the quadrupoles used for the comparison have been already built and successfully tested in particle accelerators such as the Intersecting Storage Ring [2], the Tevatron [3], the Large Electron Positron [4], HERA [5], the Relativistic Ion Collider [6], and the Large Hadron Collider [7]. The magnets named MQY 90/100/110 [8] and MQXC [9] have been designed in the framework of the LHC luminosity upgrade phase I (Nb-Ti), while those named HQ1, HQ2 [10], 90mm 2/4 layers [11] and IRQ 90/100/110 [12] have been designed for the phase II (Nb_3Sn). The TQ magnet [13] is a Nb_3Sn short model quadrupole build in the framework of the US LHC Accelerator Research Program (LARP). The quadrupole lay-outs named AP 50/100/150/200mm have been designed for a field quality study in superconducting quardrupoles [14]. LHC MQ+1 and LHC MQ+2 denote an LHC MQ coil with one or two additional layers, sketched to analyse coils with very large w/R_1. Grading has been removed from all graded magnets.

Compared to the numerical computation, the stored energy determined analytically using (10) and substituting w/R_1 by (12) agrees within 10% for all quadrupoles and within 5% for 2/3 of them (See Fig. 3).

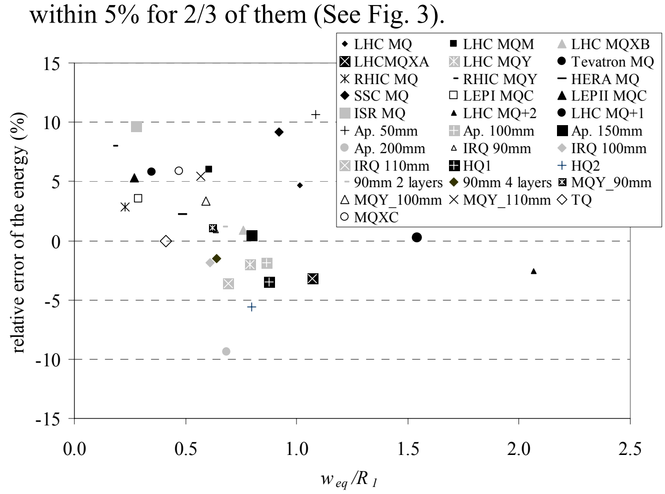

Figure 3: Agreement between numerical estimate of magnetic energy and analytical estimate for a 30º sector coil based on (10).

Effect of Grading on the Energy

The grading method consists in setting a higher current density in the outer layer in order to enhance the magnet performance. In this section we derive a heuristic coefficient to take into account the grading effect on the stored energy. Let us call j_1 and j_2 the current densities flowing in the inner and outer layers respectively. The grading parameter g is defined by:

$$g=\frac{j_2}{j_1}\qquad g>1 \tag{13}$$

The average current density j flowing through the overall cross-sectional area $S=S_1+S_2$ is

$$j=\frac{j_1S_1+j_2S_2}{S}, \tag{14}$$

where S_1 and S_2 are the area of the inner and outer layer respectively. We express j as a function of g:

$$j = j_1\left[1+\frac{S_2}{S}(g-1)\right] \qquad (15)$$

Since the magnetic energy is proportional to j^2, we expect that the variation caused by current grading is proportional to:

$$x \equiv \left[1+\frac{S_2}{S}(g-1)\right]^2 \qquad (16)$$

In Fig. 4, by means of a numerical study [1], we show that it is the case.

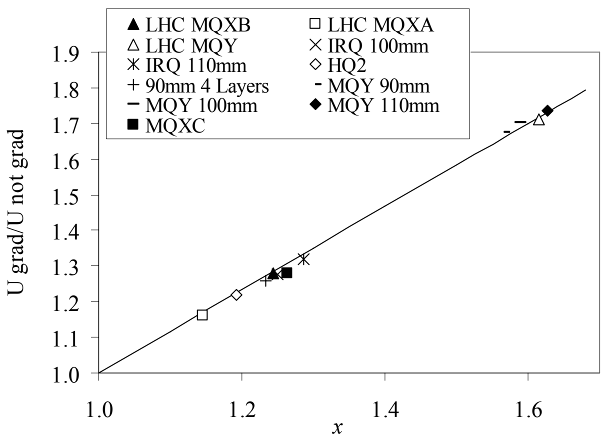

Figure 4: Increase of magnetic energy due to grading as a function of x as defined in (16).

The numerical study leading to the Fig. 3 allows us to derive the following coefficient

$$k_g = \frac{U_{grad}}{U_2} = \left[d(x-1)+1\right]. \qquad (17)$$

U_{grad} is the energy stored in the graded quadrupole, U_2 is the stored energy when the grading is removed (10), and d=1.167 is derived from the fit of the line.

Effect of Iron on the Magnetic Energy

In this section we aim at estimating the energy enhancement produce by the iron yoke through a simple analytical formula. To make it possible we assume an unsaturated iron yoke so as to use the imaging current method applied to the main harmonic of current i.e n=2. The additional stored energy U_i due to the yoke is obtained by integrating the scalar product between the vector potential produced by the yoke and the current flowing inside the coils, all over the coil cross-sectional area:

$$U_i = \frac{\pi}{128}\mu_0 {j_0}^2 {a_2}^2 {R_1}^4 \left(\frac{R_1}{R_i}\right)^4 \left[(1+t)^4-1\right]^2 \qquad (18)$$

where R_i is the inner radius of the yoke. Now we express the ratio between the total stored energy $U_{tot}=U_i+U_2$ and the ironless energy U_2 (10)

$$k_i = \frac{U_{tot}}{U_2} = 1+\frac{1}{8}\left(\frac{R_1}{R_i}\right)^4 \frac{\left[(1+t)^4-1\right]^2}{\frac{1}{4}\left[(1+t)^4-1\right]-\ln(1+t)} \qquad (19)$$

In the range of $t=w/R_1$ used on accelerator magnets, one fits the previous expression with:

$$k_i = 1+\left(\frac{R_1}{R_f}\right)^4 \left(at^2+bt+c\right) \qquad (20)$$

with a=8.52, b=-0.80 and c=1.56.

Total Formula

The energy stored in a quadrupole magnet graded or not, and with or without iron yoke can be estimated analytically with:

$$U_{all} = k_g k_i U_2 \qquad (21)$$

where U_2 is given by (10), k_g by (17) and k_i by (20). The agreement with numerical codes is within ~10%.

CONCLUSION

We used the main harmonic of the Fourier decomposition to set up a simple formula allowing to estimate the stored energy in superconducting quadrupoles made of sector coils as a function of the coil width and of the aperture. The equivalent coil width allowed us to compare the energy estimate with numerical computations in case of real coil layouts. The stored energy enhancement due to current grading or due to the presence of an iron yoke is taken into account by means of a heuristic coefficient and an analytical formula based on the imaging current method. A comparison between the stored energy computed analytically and numerically in case of real coil layout (with grading and real iron yoke) showed agreement within 10%.

REFERENCES

[1] S. Russenschuck, ed., *CERN* 99-01 (1999).
[2] J. Billan et al., CERN Yellow Report 76-16 (1976).
[3] W. E. Cooper et al., Fermilab Report TM-1183 (1983).
[4] P. J. Ferry et al., CERN LEP-MA 89-67 (1989).
[5] S. Wolff, ed. by C. Marinucci and P. Waymuth (Zurich, SIN, 1995).
[6] M. Anerella et al., *Nucl. Instrum. Meth.* A499 (2003) 280-315.
[7] O. Bruning, N. Catalan-Lasheras, and G. Kirby, CERN 2004-003 (2004).
[8] R. Ostojic et al, *PAC* (2005) 2795-2797.
[9] F. Borgnolutti, E. Todesco, *PAC* (2007) 350-352.
[10] G.L. Sabbi et al., 17 (2007), No. 2, pp 1051-1054.
[11] P. Ferracin et al., *PAC*, (2003) 1984-1986.
[12] A.V. Zlobin et al., *PAC* (2003) 1975-1977.
[13] R.C. Bossert et al., *IEEE Trans. Appl. Superconduct.*, 16 (2006), No. 2, pp 370-373.
[14] B. Bellesia et al., *LHC Project Report* 1010 (2007).