APPLICATION OF FOKKER-PLANCK EQUATION IN THE RESEARCH OF BEAM LONGITUDINAL INSTABILITY

Ge Jun, Tsinghua University, Beijing, China

Abstract

We introduce the research status of the beam longitudinal instability based on the mode coupling theory. After compared the different work we derive the Fokker-Planck equation from the equation of particle motion, and try to obtain the solution of the distribution equation including the potential well distortion.

1 INTRODUCTION

The longitudinal microwave instability, which leads to the growth of the bunch length and the increase of energy spread of the beam, is one of the main factors that determine the ultimate performance of accelerators[1]. In this paper, we introduce the present status of researching longitudinal microwave instability, and the subject of longitudinal microwave instability of a bunched beam in a circular accelerator is systematically studied with perturbation approach.

Mode coupling theory, which is the most important theory to solve the beam instability, was introduced in 70's by F. Sacherer in CERN[2], who built the famous Sacherer integral eqution. Based on the mode coupling theory, A. Chao in SLAC introduced the scaling law. In 1990, Yokoya and Oide of KEK considered that the potential well distortion in the beam stationary distribution can not be omitted[3][4].

In this paper the Fokker-Planck equation is used to describe the beam distribution. The stationary solution of the equation, using action-angle variable (J,ϕ), the matrix form which in term of generalized Laguerre polynomials and power series are partially derived from linearized Fokker-Planck equation including potential well distortion.

In section 2, we introduce the status of researching longitudinal instability. In section 3, we derive the density distribution equation for synchrotron motion based on the equation of single particle motion. In section 4, we try to solve the equation including the potential well distortion. Generally the stochastic equation is very complicated, it is hard to get the analytical solution, so we have to use numerical simulation method. Conclusions are given in section 5.

2 RESEARCH STATUS

In 60's, Vlasov equation, which describes the beam distribution in phase space, was firstly introduced to the theory of coast beam collective instability. In 70's, F. Scherer developed the theory of bunch instability systematically. But the new theory and method are needed to explain the inside mechanism, especially for the new characteristic of instability, such as sawtooth instability etc[5][6].

The level of electron beam current in modern accelerators increases quickly years by years. The synchronous radiation effect and the stochastic quantum excitation effect begin to affect the beam dynamics and lead to beam instability. The beam-environment, including the radiation damping and quantum excitation, is a non Hamilton system, and a nonlinear system[7]. To research this system helps us to understand the inherence of instability. So the Fokker-Planck equation has been used more and more widely.

People use different expansion methods to establish the synchrotron modes[8][9], but there are several problems to be solved, for example, the convergence of various polynomials. Although the effect of potential well distortion has been included in the recent years, the mechanism of the instability can not been completely discovered, either.

Also people try other methods to solve the Fokker-Planck equation, like the moments expansion[10], direct numerous calculation of the equation[11], etc.

3 FOKKER-PLANCK EQUATION

We derive the Fokker-Planck equation of longitudinal beam distribution. The complete equations of longitudinal single particle motion are:

$$\frac{d\delta}{ds}=\frac{e\hat{V}_{rf}}{CE_0}\sin\left(\omega_{rf}\frac{z}{c}+\phi_s\right)-\frac{U_0}{CE_0}-\frac{2}{c\tau_s}\delta-\frac{eV(z)}{CE_0}+\Gamma(s), \quad (1)$$

$$\frac{dz}{ds}=-\eta\delta\ . \quad (2)$$

In the case of small amplitude of synchrotron motion, these equations are simplified to:

$$\frac{d\delta}{ds}=\frac{1}{\eta}\left(\frac{\omega_s}{c}\right)^2 z-\frac{2}{c\tau_s}\delta-\frac{eV(z)}{CE_0}+\Gamma(s), \quad (3)$$

$$\frac{dz}{ds} = -\eta\delta, \tag{4}$$

where

$$\eta = \frac{1}{\beta^2}\left(\alpha + \frac{1}{\gamma^2}\right), \quad \delta = \frac{\Delta E}{E_0}. \tag{5}$$

In above five equations s means path of the particle, and δ is energy spread, η slippage factor, α momentum compaction factor, β relativistic factor, γ relativistic factor, ω_s synchrotron angular frequency, c light velocity, z position relative to the synchronous particle, τ_s damping coefficient, $V(z)$ wake potential, C circumference of accelerator, ΔE energy derivation, E_0 energy of synchronous particle, $\Gamma(s)$ stochastic function denoting the effect of quantum emission due to synchrotron radiation.

The equations of particle motion are typical two dimension Langevin equations. We can derive Fokker-Planck equation by the transition formula from Langevin equations:

$$\frac{\partial\psi}{\partial s} - \eta\delta\frac{\partial\psi}{\partial z} + \frac{\omega_s^2}{\eta c^2} z\frac{\partial\psi}{\partial\delta} - \frac{eV(z)}{CE_0}\frac{\partial\psi}{\partial\delta} = \frac{2}{c\tau_s}\frac{\partial(\delta\psi)}{\partial\delta} + D\frac{\partial^2\psi}{\partial\delta^2}, \tag{6}$$

where ψ is beam distribution function, D is a diffusion constant. If the beam damp is large enough, the right side of equation can be omitted. Then the Fokker-Planck equation becomes to Vlasov equation.

4 SOLUTION OF FOKKER-PLANCK EQUATION

4.1 Stationary Solution

If we neglect the wake field force eV due to the self-fields, the stationary solution can be found as:

$$\psi_0(z,\delta) = C'\exp\left(-\frac{z^2}{2\sigma_z^2}\right)\exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right), \tag{7}$$

where

$$\sigma_z^2 = \frac{c\tau_s D}{2}\left(\frac{\eta c}{\omega_s}\right)^2 = \left(\frac{\eta c}{\omega_s}\right)^2\sigma_\delta^2. \tag{8}$$

Here σ_z is rms bunch length, σ_δ is energy spread. C' is a normalization constant given by

$$\int\psi_0(z,\delta)dzd\delta = Ne. \tag{9}$$

N is the number of electrons in a bunch. It is obvious that the stationary distribution of beam is bi-Gaussian form.

According to Yokoya and Oide's theory, the effect of potential well distortion can not be omitted. The wake potential $V(z)$ in equation (6) is composed of two parts, $V_0(z)$ is self-consistent with ψ_0 is the stationary distribution of the beam, and $V_1(z,s)$ is self-consistent with the perturbation distribution $\psi_1\exp(-i\Omega s/c)$, Ω is mode frequency. So $V(z)$ can be written by

$$V(z) = V_0(z) + V_1(z,s), \tag{10}$$

$$V_0(z) = e\int_z^\infty dz'\rho_0(z')W_0^{//}(z'-z), \tag{11}$$

$$V_1(z,s) = e\int_z^\infty dz'\rho_1(z')\exp\left(-i\Omega\frac{s}{c}\right)W_0^{//}(z'-z). \tag{12}$$

ρ is the longitudinal density of beam. By considering the multiple turns effect and property of the longitudinal wake field $W_0^{//}(z>0)=0$, equation (12) can be expressed by use of the longitudinal coupling impedance $Z_0^{//}$ by:

$$\begin{aligned} V_1(z,s) &= e\int_{-\infty}^\infty dz'\sum_{k=-\infty}^{\infty}\rho_1(z')\exp\left(-i\Omega\frac{s}{c} - kT_0\right)W_0^{//}(z'-z) \\ &= \frac{e}{T_0}e^{-i\Omega s/c}\sum_{p=-\infty}^{\infty}\tilde{\rho}_1(p\omega_0+\Omega)e^{i(p\omega_0+\Omega)z/c}Z_0^{//}(p\omega_0+\Omega), \end{aligned} \tag{13}$$

Substituting equation (10), (11) into equation (6) we get another stationary solution including the potential well distortion, called Haïssinski equation:

$$\psi_0(z,\delta) = C''\exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right)\exp\left(-\frac{z^2}{2\sigma_z^2} + \frac{r_0}{\eta\sigma_\delta^2\gamma C}\int_0^\infty dz''\int_{z''}^\infty dz'\rho(z')W_0^{//}(z''-z')\right). \tag{14}$$

C'' In equation (14) is the new nornalization constant. The information of the wake field is included in second exponential function in equation (10). We can only get the solution by the numerical method.

4.2 Perturbation Solution

In order to obtain the perturbation solution we decompose the distribution ψ into a stationary part and perturbed part as:

$$\psi(r,\phi,s) = \psi_0(r) + \psi_1(r,\phi)\exp\left(-i\Omega\frac{s}{c}\right), \tag{15}$$

ψ_0 in Eq. (11) is the solution of Eq. (13).

In section 2 we learn some different ways to expand the distribution function ψ. Now we select the power series and Laguerre polynomial expansion in the action-angle coordinate system,

$$\psi_1(r,\phi)=\sum_{l=-\infty}^{\infty}R_l(r)e^{il\phi}\ , \tag{16}$$

$$R_l(r)=\exp\left(-\frac{r^2}{2\sigma_z^2}\right)\sum_{k=0}^{\infty}a_k^l\, f_k^l(r)\,, \tag{17}$$

where

$$f_k^l(r)=\sqrt{\frac{k!}{(l+k)!}}r^{l/2}L_k^l(r), \tag{18}$$

$L_k^l(r)$ are generalized Laguerre polynomials, $f_k^l(r)$ satisfy an orthogonality relation,

$$\int_0^\infty \exp\left(-\frac{r^2}{2\sigma_z^2}\right)f_k^l(r)f_m^l(r)dx=\delta_{km}\,. \tag{19}$$

After substituting Eq. (14)-Eq. (19) into Eq. (6), Eq. (6) is transformed into a matrix equation. By computing the eigenvalue we can get the current threshold value and calculate the mode frequency. The study is on the way continuously.

5 CONCLUSION

Based on the mode coupling theory, different methods have been used on the Fokker-Planck equation. The polynomials expansion is the traditional one[12]. We introduce the research status of the equation, and one of the methods including the potential well distortion, but the computation is not involved in this paper. Our work will focus on the convergence of the Laguerre polynomial expanding the Fokker-Plank equation which including the potential well distortion, and try to find new methods to solve this equation. Also we hope to calculate the threshold current of BEPC more accurately by Fokker-Planck equation. These works are in progress.

ACKNOWLEDGE

Thanks to Professor Guo Zhiyuan of IHEP, he gives me some advisable suggestion on my work and read this paper, also thanks to professor Lin Yuzheng of Tsinghua University for some discussion.

REFERENCES

[1] A. Chao, Physics of Collective Beam Instabilities in High Energy Accelerators, John Wiley & Sons, New York, (1993)

[2] F. Sacherer, CERN SI-BR/72-5, 1972

[3] K. Oide, K. Yokoya, Longitudinal Single-Bunch Instability in Electron Storage Rings, KEK Preprint 90-10 (1990)

[4] K. Oide, A Mechanism of Longitudinal Single-Bunch Instability in Storage Rings, KEK-94-138 (1994)

[5] K. Bane, et al., High-Intensity Single Bunch Instability Behavior in the New SLC Damping Ring Vacuum Chamber, PAC95, 5, 3109

[6] B. V. Podobedov, R. H. Siemann, Sawtooth Instability Studies in the Stanford Linear Collider Damping Rings, PAC97, 2, 1629

[7] M. Sands, The Physics of Electron Storage Rings and Introduction, SLAC-121

[8] T. Suzuki, Part. Accel. 14, 91 (1983)

[9] Chen Bo, Longitudinal Collective Instability of Nonlinear Hamiltonian Systems in a Circular Accelerator, Ph.D thesis, SLAC (1995)

[10] S. Heifets, A. Chao, Study of Microwave Instability, SLAC-PUB-7390 (1996)

[11] M. P. Zorzano, H. Mais, Numerical Solution for Fokker-Planck Equation in Accelerators, PAC97, 2, 1825

[12] H. Risken, Fokker-Planck Equation, Springer, Berlin, 1989