The Emittance Growth Scaling Laws in Resonance Crossing

X. Pang, F. Wang, X. Wang, S.Y. Lee
IUCF, Indiana University
K.Y. Ng
Fermilab

Introduction

- FFAG → a favorable candidate for high power proton driver.
  - Constant guide field → rep rate can be very high, in the kHzs.
  - Scaling design → nonlinear fields, large magnet apertures.

    → fixed tune, but fringing field or flux saturation in iron poles can cause tune change.
  - Non-scaling design → linear fields, much smaller magnet apertures.

    →tunes change by many units in a ramp cycle.
- Beam quality can deteriorate when crossing resonances.
- Tune-ramp rate

$$\frac{\Delta \nu_{x,z}}{\Delta n} \approx -\frac{\nu_{x,z}}{\beta^2 E}\frac{\Delta E}{\Delta n}$$

typically ~ -10^{-4} to $-5*10^{-3}$ per turn

Example

- Ruggiero suggested 3 concentric FFAGs as a proton driver to replace BNL AGS to reach 10 MW beam power.
- Tunes change from

  $(\nu_x, \nu_z) = (40,38.1)$ to $(19.1,9.3)$
- Cross systematic 4^{th} and 6^{th}

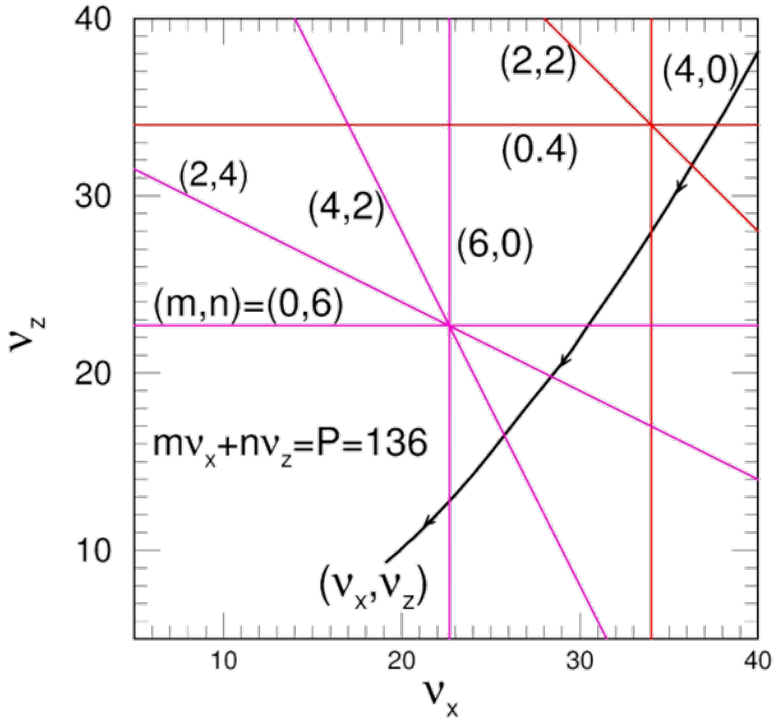
  resonances: (P=136)

  $4\nu_x=P,\ 4\nu_z=P,\ 2\nu_x+2\nu_z=P$

  $6\nu_x=P,\ 6\nu_z=P,\ 2\nu_x+4\nu_z=P,\ 4\nu_x+2\nu_z=P$
- S.Y. Lee pointed out emittance

  growth can be large if crossing

  rate is slow, and gave a scaling relationship.



- Thus phase advance per cell cannot be near 90^0 and 60^0.
- Lattice design can become very restricted.

The Model

- Lee, et al. studied sp-ch driven 4th order systematic resonances and field-error driven linear resonances.
- We study here sp-ch driven 6th order systematic resonances and octupole driven 4th order parametric resonances.
- Our study bases on simulations.
- Lattice is similar to Fermilab Booster, with P = 24 FODO cells.
- Sp-ch kicks applied at the center of D and F quadrupoles, two kicks per superperiod. (Making 4 kicks per superperiod does not change our results)
- Transport matrices used from magnet to magnet.
- Kinetic energy fixed at 1 GeV, tunes allowed to ramp.
- Syn. oscillation neglected since emittance usually grows much faster.
- Assume bi-Gaussian distribution: $\rho(x,z) = \frac{Ne}{2\pi\sigma_x\sigma_z} e^{-x^2/2\sigma_x^2 - z^2/2\sigma_z^2}$

Systematic Resonances

- Sp Ch potential: $V_{sc}(x,z) = \frac{K_{sc}}{2}\int_0^\infty \frac{\exp\left[-\frac{x^2}{2\sigma_x^2+t}-\frac{z^2}{2\sigma_z^2+t}\right]-1}{\sqrt{(2\sigma_x^2+t)(2\sigma_z^2+t)}}\,dt, \qquad K_{sc} = \frac{2Nr_0}{\beta^2\gamma^3}$

- Taylor's Expansion: $(r = \sigma_z/\sigma_x)$

$$
\begin{aligned}
V_{sc}(x,z) = -\frac{K_{sc}}{2}\Bigg\{ & \left[\frac{x^2}{\sigma_x(\sigma_x+\sigma_z)}+\frac{z^2}{\sigma_z(\sigma_x+\sigma_z)}\right] \\
& -\frac{1}{4\sigma_x^2(\sigma_x+\sigma_z)^2}\left[\frac{2+r}{3}x^4+\frac{2}{r}x^2z^2+\frac{1+2r}{3r^3}z^4\right] \\
& +\frac{1}{72\sigma_x^3(\sigma_x+\sigma_z)^3}\left[\frac{8+9r+3r^2}{5}x^6+\frac{3(3+r)}{r}x^4z^2\right. \\
& \left.+\frac{3(3r+1)}{r^3}x^2z^4+\frac{8r^2+9r+3}{5r^5}z^6\right]+\cdots\Bigg\},
\end{aligned}
$$

4th order

6th order

- Each particle passing through a length Δs experiences a space-charge kick:

$$
\begin{aligned}
\frac{\Delta x'}{\Delta s} &= -\frac{\partial V_{sc}}{\partial x} = F_{x,sc}, \\
\frac{\Delta z'}{\Delta s} &= -\frac{\partial V_{sc}}{\partial z} = F_{z,sc}.
\end{aligned}
$$

Space Charge Force

- In our simulation:
  - not round beam → complex error function
  - round beam → power series

✓ If the beam is not round beam, it is straightforward to show

$$\frac{\Delta x'}{\Delta s} - j\frac{\Delta z'}{\Delta s} = \frac{K_{sc}}{2}\hat{F}_{sc},$$

$$\hat{F}_{sc} = j\frac{\sqrt{2\pi}}{\sqrt{(\sigma_x^2 - \sigma_z^2)}}[w(a+jb) - e^{-(a+jb)^2+(ar+j\frac{b}{r})^2}\, w(ar + j\frac{b}{r})],$$

Where $w(Z) = e^{-Z^2}[1 + \frac{2j}{\sqrt{\pi}}\int_0^Z e^{\zeta^2} d\zeta],$

singularity when r=1

$$a = \frac{x}{\sqrt{2(\sigma_x^2 - \sigma_z^2)}}, b = \frac{z}{\sqrt{2(\sigma_x^2 - \sigma_z^2)}}, \; v^2 = \frac{2\sigma_z^2 + t}{2\sigma_x^2 + t}, \; r = \frac{\sigma_z}{\sigma_x}.$$

Space Charge Force

- ✓ For round beam, we need to expand the space-charge force around $\sigma_x=\sigma_z$ or r=1.

$$F_{x,sc} = -\frac{\partial V_{sc}}{\partial x} = K_{sc}\frac{x}{\sigma_x^{\,2}} f_x(x^2, z^2)$$

$$F_{z,sc} = -\frac{\partial V_{sc}}{\partial z} = K_{sc}\frac{z}{\sigma_z^{\,2}} f_z(x^2, z^2)$$

$$\varepsilon = 1 - r = \frac{\sigma_x - \sigma_z}{\sigma_x}$$

$$f_x(x^2, z^2) = \frac{1}{1-r^2}\int_r^1 dv\, e^{-a^2(1-v^2)-b^2\left(\frac{1}{v^2}-1\right)}$$

$$f_z(x^2, z^2) = \frac{r^2}{1-r^2}\int_r^1 \frac{dv}{v^2}\, e^{-a^2(1-v^2)-b^2\left(\frac{1}{v^2}-1\right)}$$

where $a = \frac{x}{\sqrt{2(\sigma_x^2-\sigma_z^2)}}, b = \frac{z}{\sqrt{2(\sigma_x^2-\sigma_z^2)}}, \; v^2 = \frac{2\sigma_z^2+t}{2\sigma_x^2+t}$

$$f_x(x^2, z^2) = \frac{\sigma_x}{\sigma_x+\sigma_z}\int_0^1 dt \exp\left[-\frac{x^2 t(1-\varepsilon t/2)}{\sigma_x(\sigma_x+\sigma_z)} - \frac{z^2 t(1-\varepsilon t/2)}{\sigma_x(\sigma_x+\sigma_z)(1-\varepsilon t)^2}\right], t = (1-v)/\varepsilon$$

Space Charge Force

- Do Taylor's Expansion and using the integral:

$$f_n(w)=\int_0^1 e^{-wt}t^n dt=\frac{n!}{w^{n+1}}(1-e^{-w}\sum_{k=0}^{n}\frac{w^k}{k!}2),\ w=p+q,\quad p=\frac{x^2}{\sigma_x(\sigma_x+\sigma_z)},\ q=\frac{z^2}{\sigma_x(\sigma_x+\sigma_z)}$$

We obtain:

$$f_x(x^2,z^2)=\frac{\sigma_x}{\sigma_x+\sigma_z}\{f_0(w)+\varepsilon\frac{p-3q}{2}f_2(w)+\varepsilon^2[-2qf_3(w)+\frac{(p-3q)^2}{8}f_4(w)]$$
$$+\varepsilon^3[-\frac{5q}{2}f_4(w)-(p-3q)qf_5(w)+\frac{(p-3q)^3}{48}f_6(w)]$$
$$+\varepsilon^4[-3qf_5(w)-\frac{5pq-23q^2}{4}f_6(w)-\frac{(p-3q)^2q}{4}f_7(w)-\frac{(p-3q)^4}{384}f_8(w)]$$
$$+\varepsilon^5[-\frac{7}{2}qf_6(w)-\frac{3pq-19q^2}{2}f_7(w)-\frac{5p^2q-46pq^2+93q^3}{16}f_8(w)]$$
$$-\frac{(p-3q)^3q}{24}f_9(w)+\frac{(p-3q)^5}{3840}f_{10}(w)]+O(\varepsilon^6)\}.$$

When particle resides at the center of the beam $w=0$.
Whenever w is small, use

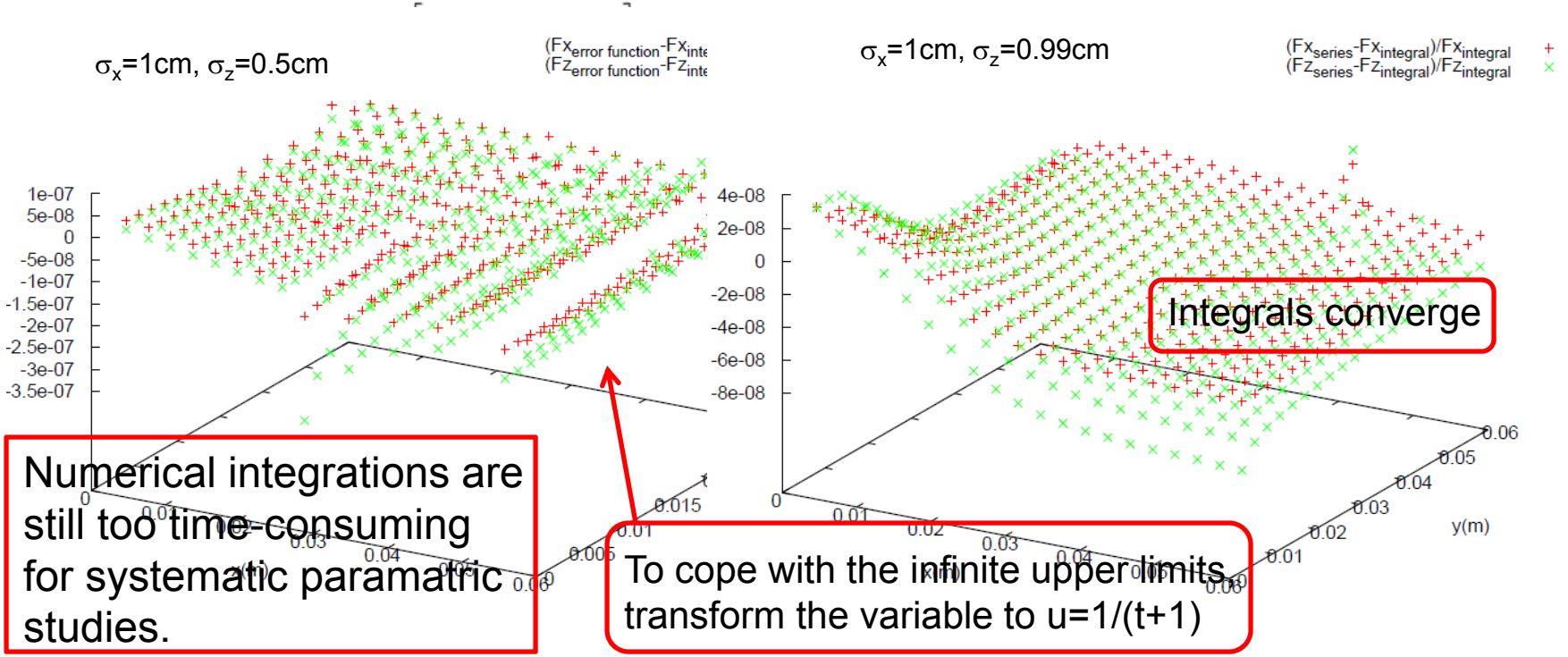
$$f_n(w)=\sum_{k=0}^{\infty}\frac{(-w)^k}{(n+k+1)k!}$$

to remove the singularity.

An expansion up to O(ε^3) together with $\varepsilon \le 0.01$ will provide a space charge force that is accurate up to one part in one million.

Space Charge Force

- To ensure the accuracy of our calculation of the space charge force, we compare the computed space charge force using complex error function and series with results obtained by direct numerical integration.



6th order Systematic Resonance

- In action-angle variables,

$$V_{sc,6}(J_x, J_z, \psi_x, \psi_z, \theta) \approx -\frac{1}{R}\sum_{\ell} |G_{60\ell}| J_x^3 \cos(6\psi_x - \ell\theta + \chi_{60\ell}) \quad \leftarrow 6\nu_x = P$$

$$-\frac{1}{R}\sum_{\ell} |G_{06\ell}| J_z^3 \cos(6\psi_z - \ell\theta + \chi_{06\ell}) - \cdots \quad \leftarrow 6\nu_z = P$$

$$G_{60\ell} = \frac{1}{5760\pi}\oint \frac{K_{sc}\beta_x^3(8\sigma_x^2 + 9\sigma_x\sigma_z + 3\sigma_z^2)}{\sigma_x^5(\sigma_x+\sigma_z)^3} e^{j(6\phi_x - 6\nu_x\theta + \ell\theta)} ds$$

$$G_{06\ell} = \frac{1}{5760\pi}\oint \frac{K_{sc}\beta_z^3(8\sigma_z^2 + 9\sigma_x\sigma_z + 3\sigma_x^2)}{\sigma_z^5(\sigma_x+\sigma_z)^3} e^{j(6\phi_z - 6\nu_z\theta + \ell\theta)} ds$$

- Can factor out sp-ch dependent part of resonance strength, giving dimensionless reduced strength g_{mnl}:

$$G_{mn\ell} = g_{mn\ell}\frac{K_{sc}R}{4\epsilon_{\text{rms}}^3}$$

lattice dependent

sp-ch dependent

Simulation Parameters

- Proton injection rate 4×10^{11} particles per turn.
- Harmonic number h= 84.
- Circumference C=474.2m.
- Bunching factor $B = C/(h\sqrt{2\pi}\sigma_s) = 2$
- Initial emittance: $\varepsilon_{N,rms} = 8.33\times10^{-6}\pi m$
- Kinetic energy is kept constant at 1GeV during the tracking.

Sample Simulation

- 100 turns injection
- Crossing systematic resonances:

$6\nu_x = P, \quad 6\nu_z = P, \quad P = 24$

$(\nu_{x0}, \nu_{z0}) = (4.25, 4.45)$

$\Downarrow$

$(\nu_x, \nu_z) = (3.69, 3.89)$

$d\nu_{x,z}/dn = -0.0004$

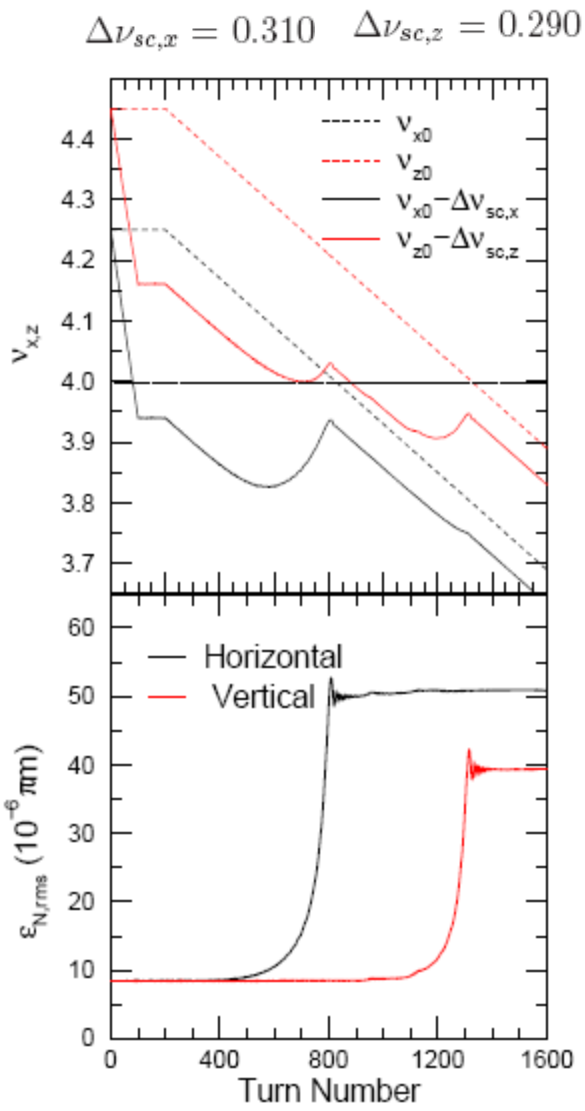
$|g_{60P}| = 0.00181$

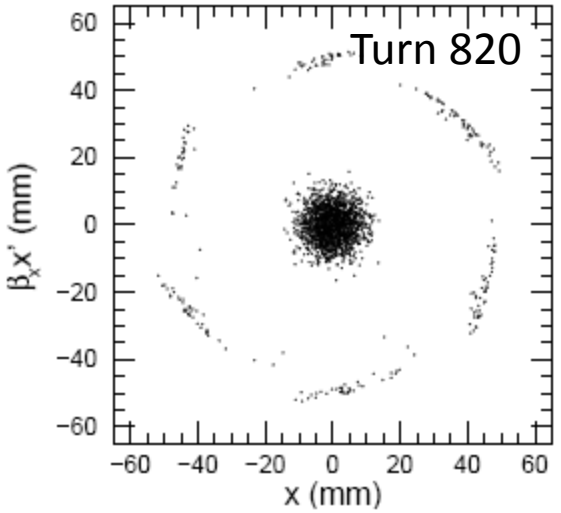
$|g_{06P}| = 0.00172$

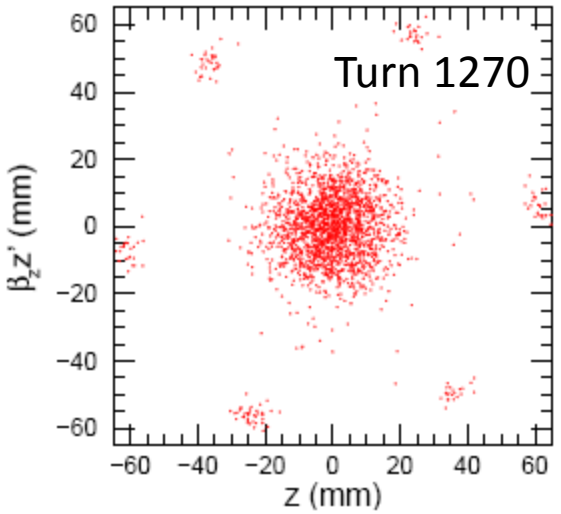
Aperture= $500\pi mm - mrad$

- Emittance growth factor (EGF):

Final emit./initial emit.



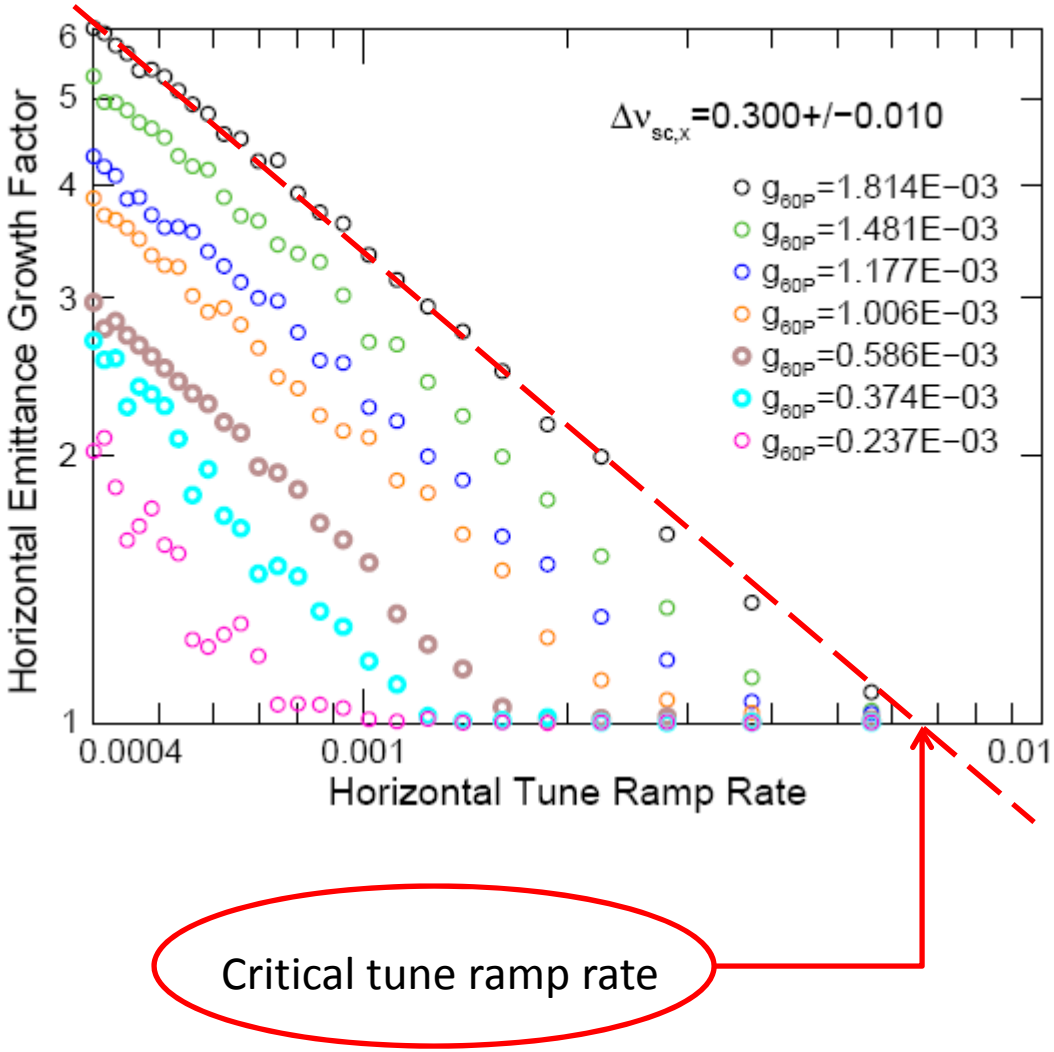




Scaling Power Laws

- Tune ramp rate:
0.0004 → 0.0112
- Log-log plots
- Scaling Law

$$\mathrm{EGF} = (-d\nu/dn)^{-a}$$



Critical Tune Ramp Rate

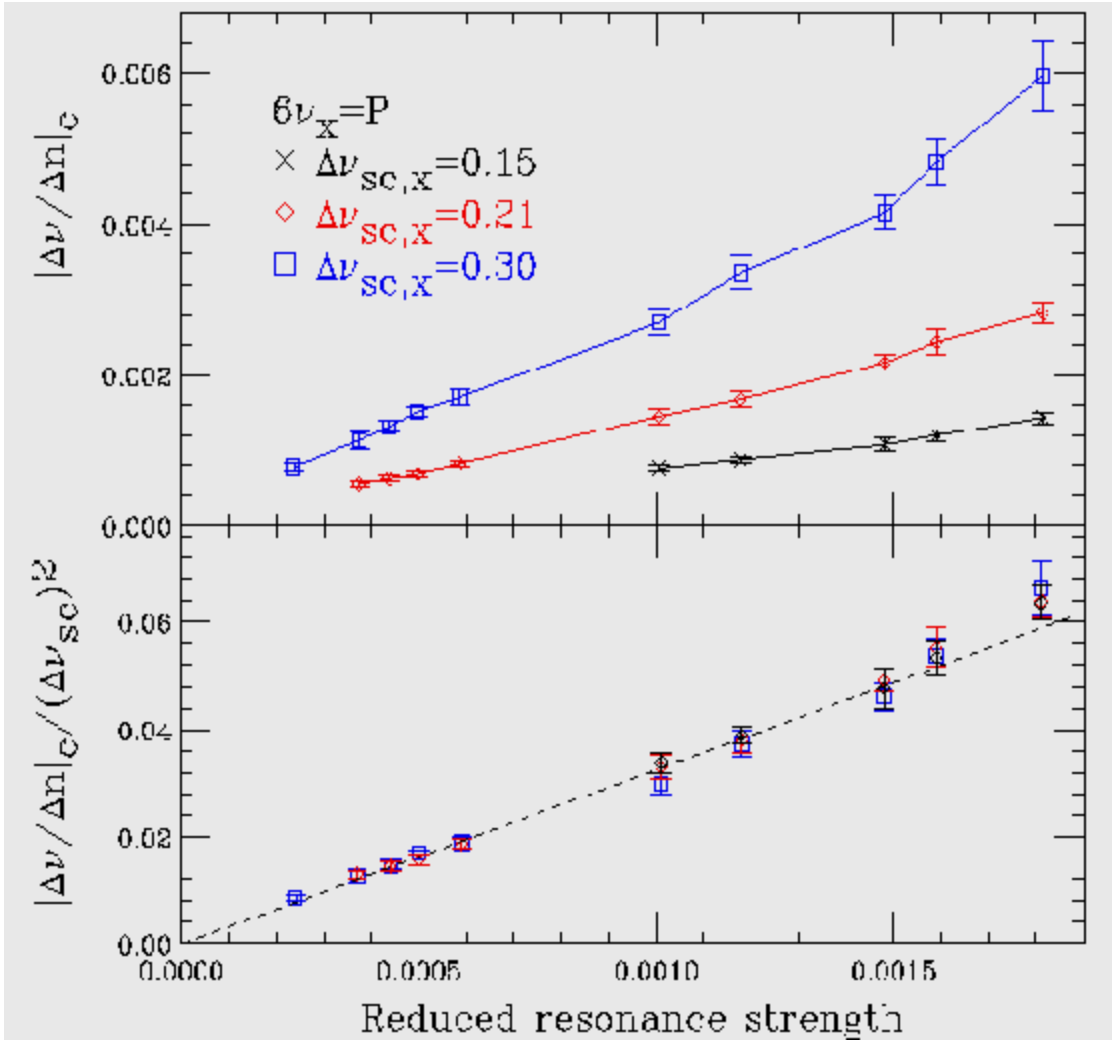
At the critical tune ramp rate, emittance is almost unchanged.

Emittance ~ G·Δn → constant
G : resonance strength
Δn: number of turn under the influence of resonance.

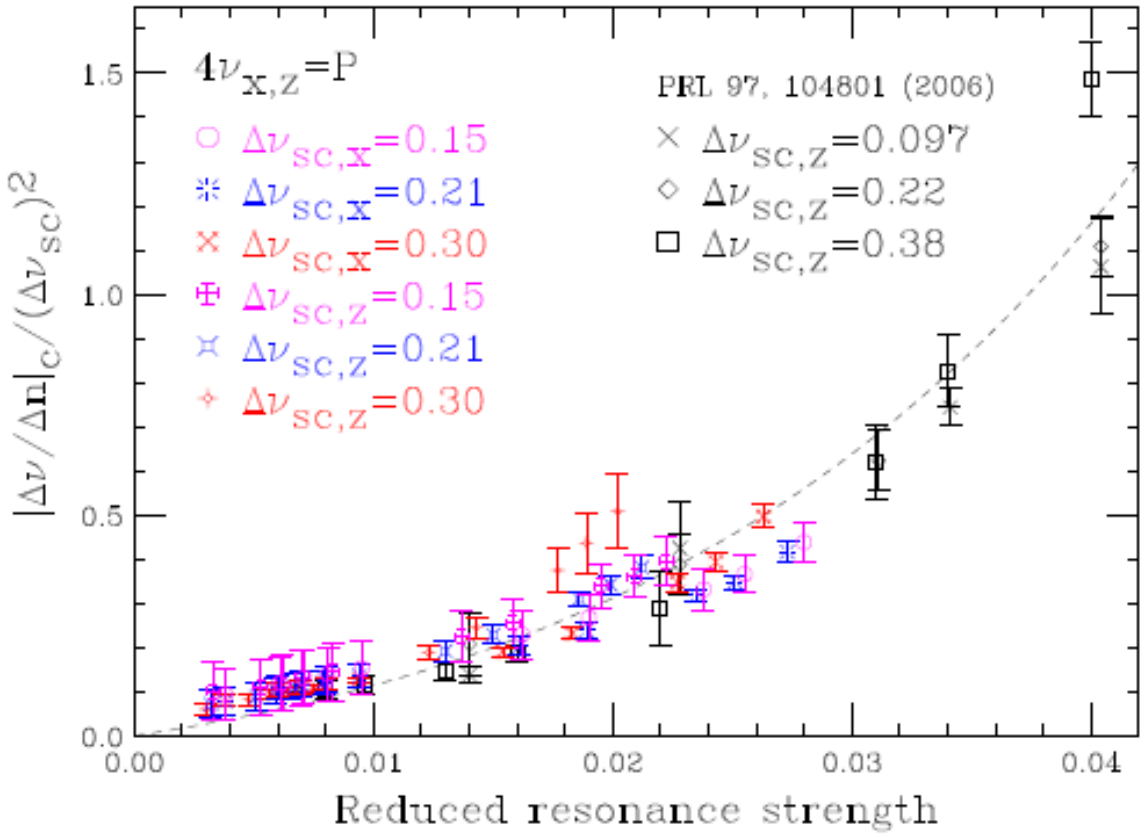
G~ g·$\Delta\nu_{sc}$
Δn ~ $\Delta\nu_{sc}$ /|dν/dn|$_c$

$$\left|\frac{d\nu}{dn}\right|_c \approx 65\ g_{60P}(\Delta\nu_{sc})^2$$

Can serve as a guideline for FFAG design!



Similar scaling law for the 4th order Systematic Resonances



$$\left|\frac{d\nu}{dn}\right|_c \approx 8.4\,(\Delta\nu_{sc})^2\, g_{04P}\, \exp\{31 g_{04P}\}.$$

Conclusion

- Power scaling laws obtained between EGF and $d\nu_z/dn$ for crossing sp-ch driven systematic 6th order resonances
- For a ring like Fermilab Booster,

with $|g_{60P}|$~0.00118, $\Delta\nu_{sc,x}$=0.30, $|d\nu_x/dn|_{crit}$~0.006/turn

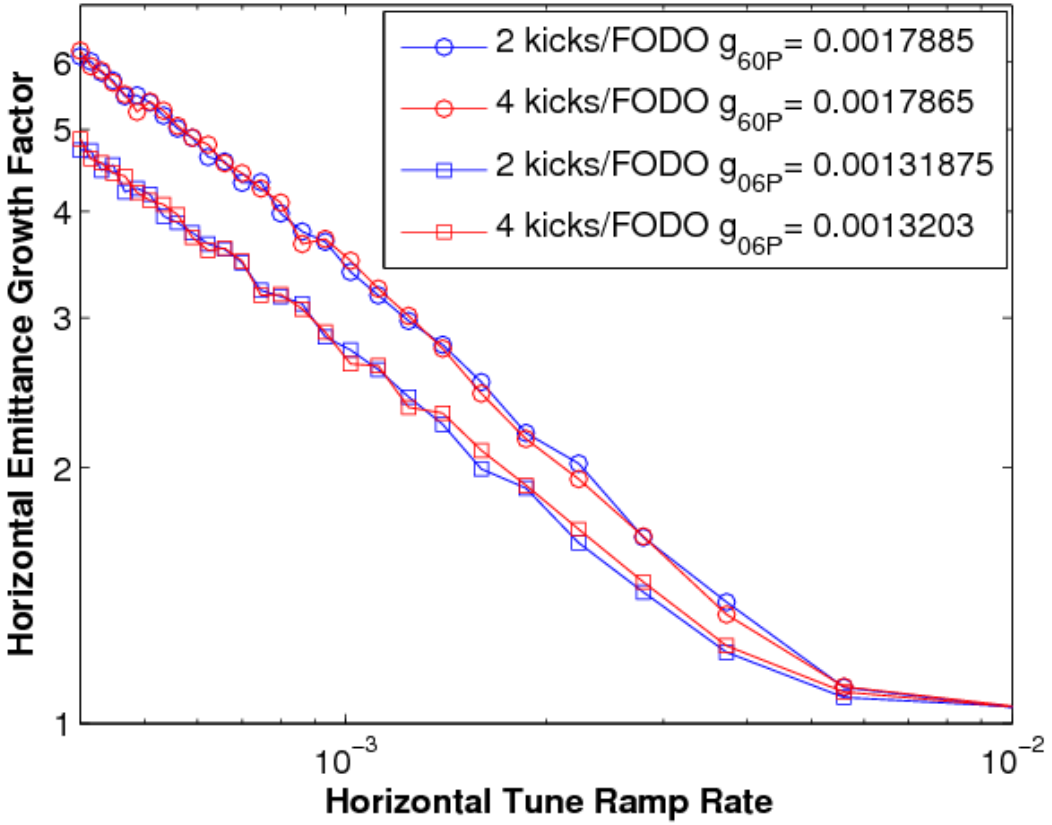
with kinetic energy 1GeV, and ν_x =4, $(dE/dn)_{crit}$ ~ 2.2MeV

- Solution:

Design a machine with smaller stopband!

What if we have more space-charge kicks

The effect of the space charge kicks should essentially depend on the harmonic content.



4th Order Parametric Resonance

- One octupole is added at D-magnet in last cell to mimic random 4th order parametric resonance.
- Potential: $V_4(x,z) = -\frac{1}{4!}\frac{B_3}{B\rho}(x^4 - 6x^2z^2 + z^4)$
- In action-angle variables:

$4\nu_x = \ell$

$$V_4(J_x, J_z, \psi_x, \psi_z, \theta) \approx -\frac{1}{R}\sum_\ell |G_{40\ell}| J_x^2 \cos(4\psi_x - \ell\theta + \chi_{40\ell})$$

$$-\frac{1}{R}\sum_\ell |G_{04\ell}| J_z^2 \cos(4\psi_z - \ell\theta + \chi_{04\ell})$$

$4\nu_z = \ell$

$$-\frac{1}{R}\sum_{\ell,\pm} |G_{2\pm2\ell}| J_x J_z \cos(2\psi_x \pm 2\psi_z - \ell\theta + \chi_{2\pm2\ell})$$

$$G_{40\ell} = \frac{1}{96\pi}\oint \frac{B_3\beta_x^2}{B\rho}\exp\left[j(4\phi_x - 4\nu_x\theta + \ell\theta)\right]ds,$$

$$G_{04\ell} = \frac{1}{96\pi}\oint \frac{B_3\beta_z^2}{B\rho}\exp\left[j(4\phi_z - 4\nu_z\theta + \ell\theta)\right]ds,$$

$$G_{2\pm2\ell} = -\frac{1}{16\pi}\oint \frac{B_3\beta_x\beta_z}{B\rho}\exp\left\{j\left[2\phi_x \pm \phi_z - (2\nu_x \pm 2\nu_z)\theta + \ell\theta\right]\right\}ds.$$

- Octupole kick: $\Delta x' = \frac{1}{6}S_4(x^3 - 3xz^2)$, $\Delta z' = \frac{1}{6}S_4(z^3 - 3x^2z)$, $S_4 = B_3\Delta s/B\rho$
- Dimensionless reduced resonant strength: $g_{mn\ell} = G_{mn\ell}\epsilon_{\mathrm{rms}}$

Simulation Parameters

- Kinetic energy 1GeV
- Octupole strength $S_4 = 50\ m^{-3}$, pole tip field 0.035T with aperture 5cm and length 1m.
- Placed at defocusing Quadrupole.
- 3×10^{11} particles are injected per turn.
- Tracking for 1200 turns. Initial tune $(\nu_{x0},\nu_{z0})=(6.975,6.875)$

Sample Simulation

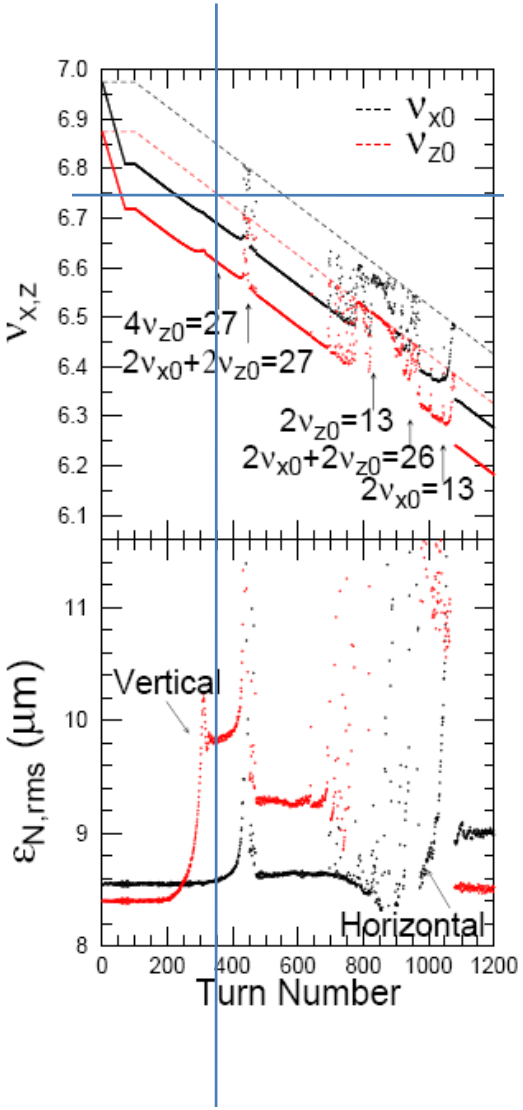
- $\beta_{x,D}$=6.3m, $\beta_{z,D}$=21.4m
- S_4=20m^{-3}
- 70 –turn injection

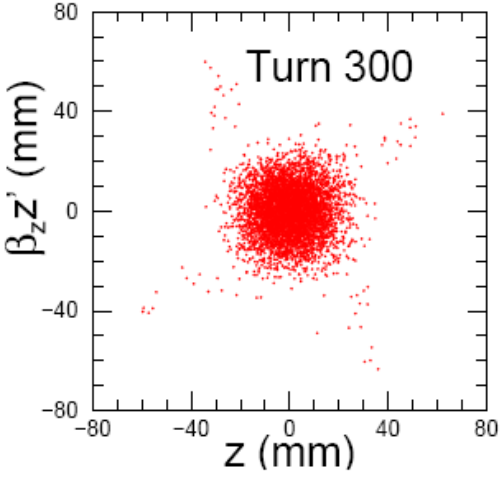
- The space charge turn shift: ($\Delta\nu_{sc,x}$, $\Delta\nu_{sc,z}$)=(0.167,0.165)

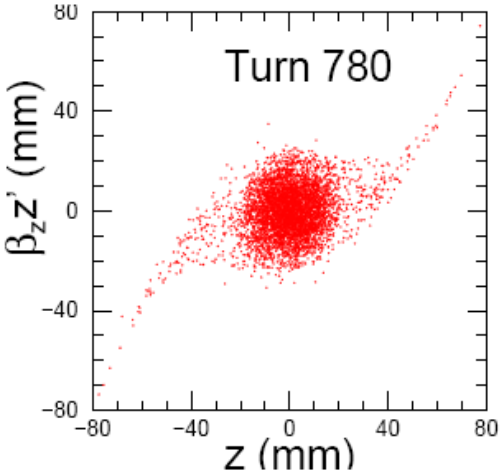
- Bare tunes are ramped down from 100th turn at a rate of:

$$|d\nu_{x0,z0}/dn| = 0.0005.$$

- Heavy beam loss is observed 6.5% during 1200 turns



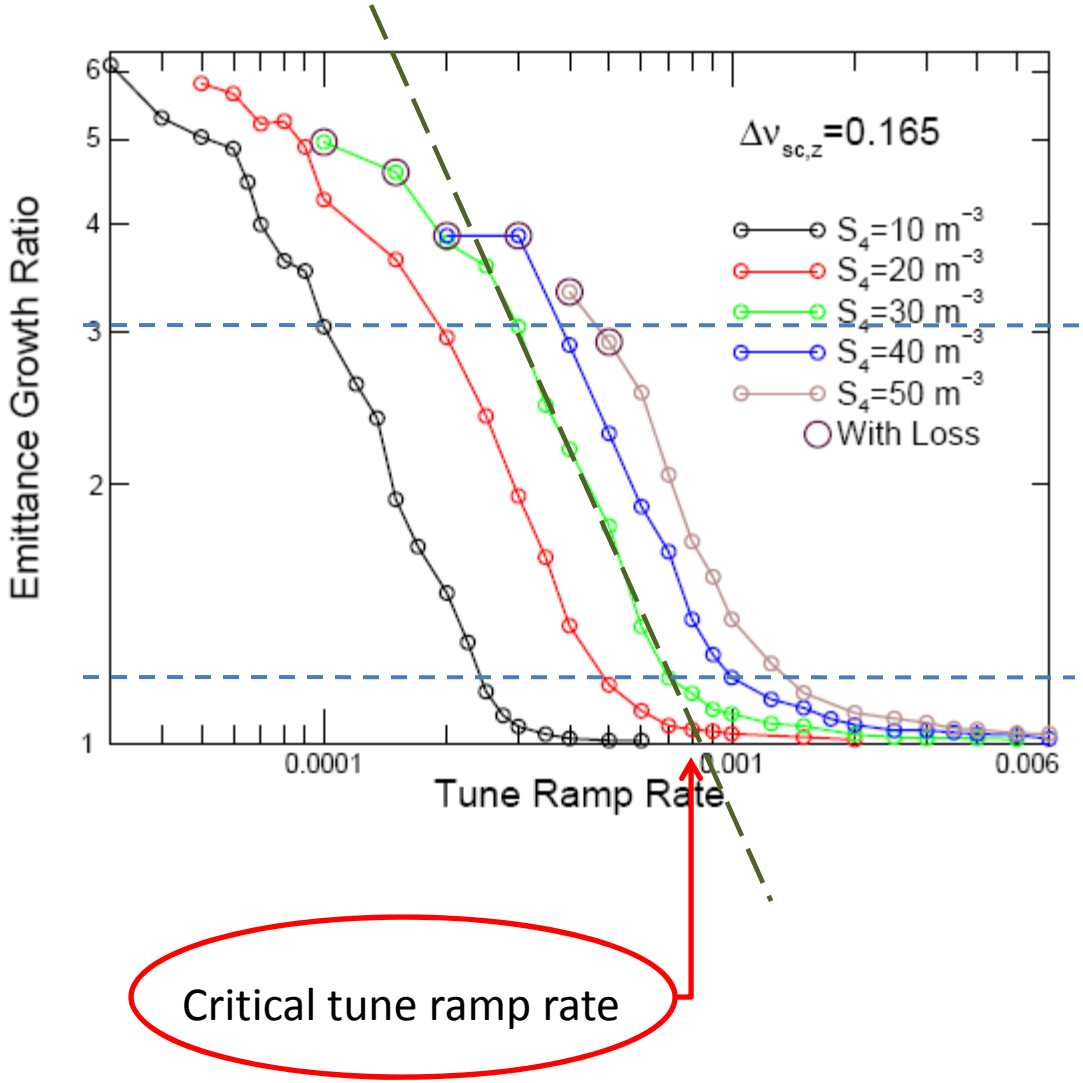




Scaling Power Laws

$$\mathrm{EGF} \sim |d\nu/dn|^{-a},$$

where $a \sim 0.8$ to 1.0



Critical Tune Ramp Rate

- Octupole driven 4th order resonance: $4\nu_z = 27$
- Given $\Delta\nu_{sc,z}$ and $|g_{04l}|$, this gives min. tune ramp rate so that EGF remains tolerable.
- It is clear that sp-ch contribution is significant.

$$\left|\frac{d\nu}{dn}\right|_c \approx 23\ g_{40\ell}\ (\Delta\nu_{sc})^{0.8}$$

